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The giant dipole vortex

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ABSTRACT
The field lines of energy flow of radiation emitted by an oscillating electric dipole in free space are either straight lines (linear dipole) or they form a vortex (rotating dipole). When the dipole is embedded in a material, the properties of the medium affect the direction of energy flow. Damping due to the imaginary part of the relative permittivity ε_r makes the field lines curve for the case of a linear dipole, and for a rotating dipole, the shape of the vortex is altered. In addition, a negative value of the real part of ε_r has the effect that the rotation direction of the vortex reverses for the case of a rotating dipole. The value of the relative permeability μ_r has in general not much effect on the redistribution of the direction of energy propagation. We show that a dramatic effect occurs when the embedding material is near-single-negative (both ε_r and μ_r approximately real, and the real parts of opposite sign). The curving of field lines is in general a sub-wavelength phenomenon. For near-single-negative materials, however, this curving extends over large distances from the dipole. In particular, the small free-space vortex of a rotating dipole becomes a vortex of enormous dimensions when the radiation is emitted into a near-single-negative material.

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1. Introduction

Electromagnetic energy travels along the field lines of the Poynting vector. Far away from the source and any obstacles, these field lines are straight and reminiscent of optical rays. Close to the source or any objects, the field lines are usually curved, provided that details of the radiation field are resolved on a scale of a wavelength and less. We shall consider the seemingly simple case of radiation emitted by an electric dipole, oscillating with angular frequency ω. The dipole moment is given by

$$\boldsymbol{d}(t) = d_o \mathrm{Re}(\hat{\boldsymbol{u}}\, e^{-i\omega t}), \tag{1}$$

with $d_o > 0$. Vector $\hat{\boldsymbol{u}}$ is complex, and we shall assume that it is normalized as $\hat{\boldsymbol{u}}^* \cdot \hat{\boldsymbol{u}} = 1$. For $\hat{\boldsymbol{u}} = \boldsymbol{e}_z$, we have a linear dipole, oscillating along the z-axis. The field lines of the Poynting vector are straight at all distances from the dipole, as shown in Figure 1. For

$$\hat{\boldsymbol{u}} = -\frac{1}{\sqrt{2}}(\boldsymbol{e}_x + i\boldsymbol{e}_y), \tag{2}$$

the dipole moment rotates over a circle in the xy plane, and the rotation is counterclockwise when viewed from the positive z-axis. The field lines swirl around the z-axis, and each field line lies on a cone. Figure 2 shows one field line. The dimensionless Cartesian coordinates are $\bar{x} = k_o x$, etc. with $k_o = \omega/c$. In this way, a distance of 2π corresponds to one free-space optical wavelength. Within about a wavelength from the dipole, the field lines form a vortex, and at larger distances from the source, they level off asymptotically to straight lines. We have named this 'the dipole vortex' [1]. Due to the rotation near the source, a field line in the far field is displaced, as compared to a field line that would come straight out of the dipole [2], and this has been observed experimentally [3]. In its most general state of oscillation, vector $\boldsymbol{d}(t)$ traces out an ellipse in a plane [4,5]. The field lines still lie on a cone, but the spatial extent of the vortex diminishes with increasing eccentricity of the ellipse [6]. In the limit where the minor axis shrinks to zero, we recover the case of a linear dipole, and the vortex disappears (as in Figure 1).

2. Embedded dipole

We shall assume that the oscillating dipole is located at the origin of coordinates and embedded in a linear

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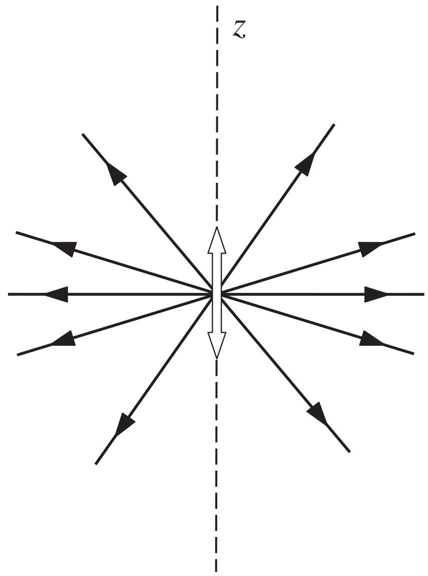

Figure 1. The figure shows the field lines of the Poynting vector for radiation emitted by an electric dipole, oscillating along the z-axis.

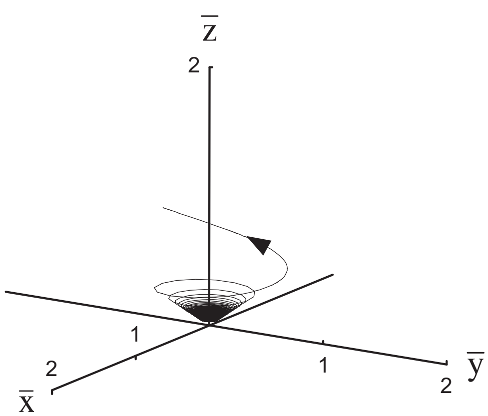

Figure 2. The field lines of the Poynting vector for an electric dipole moment that rotates in the xy-plane form an optical vortex. Shown is one field line. Close to the dipole, the field lines wind around the z-axis, and at larger distances, they go over in a straight line.

homogeneous isotropic material, characterized by the relative permittivity ε_r and the relative permeability μ_r. Both parameters are complex, in general, with non-negative imaginary parts. The index of refraction n is defined as the solution of $n^2 = \varepsilon_r\mu_r$, $\mathrm{Im}\,n \geq 0$. For ε_r and μ_r both positive or both negative, this leaves an ambiguity. By considering the limit where the imaginary parts of ε_r and μ_r go to zero, we find that we should take n as having the same sign as ε_r and μ_r [7].

When the dipole moment oscillates harmonically with angular frequency ω, so do the electric and magnetic fields. We write for the electric field

$$\boldsymbol{E}(\boldsymbol{r},t) = \mathrm{Re}[\boldsymbol{E}(\boldsymbol{r})e^{-i\omega t}], \tag{3}$$

with $\boldsymbol{E}(\boldsymbol{r})$ the complex amplitude, and the magnetic field $\boldsymbol{B}(\boldsymbol{r},t)$ is represented similarly. In order to minimize the number of parameters, we introduce the dimensionless complex amplitudes $\boldsymbol{e}(\boldsymbol{r})$ and $\boldsymbol{b}(\boldsymbol{r})$ as

$$\boldsymbol{E}(\boldsymbol{r}) = \zeta\, \boldsymbol{e}(\boldsymbol{r}), \tag{4}$$

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\zeta}{c}\boldsymbol{b}(\boldsymbol{r}), \tag{5}$$

where

$$\zeta = \frac{\mu_r d_o k_o^3}{4\pi\varepsilon_o}. \tag{6}$$

The complex amplitudes of the electric and magnetic fields are [8]

$$\boldsymbol{e}(\boldsymbol{q}) = \left\{\hat{\boldsymbol{u}} - (\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{u}})\hat{\boldsymbol{r}} + [\hat{\boldsymbol{u}} - 3(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{u}})\hat{\boldsymbol{r}}]\frac{i}{nq}\left(1+\frac{i}{nq}\right)\right\}\frac{e^{inq}}{q}. \tag{7}$$

$$\boldsymbol{b}(\boldsymbol{q}) = n(\hat{\boldsymbol{r}}\times\hat{\boldsymbol{u}})\left(1+\frac{i}{nq}\right)\frac{e^{inq}}{q}. \tag{8}$$

Here, $\boldsymbol{q} = k_o\boldsymbol{r}$ is the dimensionless field point, and $q = |\boldsymbol{q}|$. The unit vector $\hat{\boldsymbol{r}}$ from the dipole to the field point is the same as $\hat{\boldsymbol{q}}$.

3. Poynting vector

The time-averaged Poynting vector for electromagnetic radiation in a medium is given as [9]

$$\boldsymbol{S}(\boldsymbol{r}) = \frac{1}{2\mu_o}\mathrm{Re}\left(\frac{1}{\mu_r}\boldsymbol{E}(\boldsymbol{r})^*\times\boldsymbol{B}(\boldsymbol{r})\right). \tag{9}$$

The dimensionless Poynting vector $\boldsymbol{\sigma}(\boldsymbol{q})$ is defined as

$$\boldsymbol{S}(\boldsymbol{r}) = \frac{|\zeta|^2}{2\mu_o c}\boldsymbol{\sigma}(\boldsymbol{q}), \tag{10}$$

so that

$$\boldsymbol{\sigma}(\boldsymbol{q}) = \mathrm{Re}\left(\frac{1}{\mu_r}\boldsymbol{e}(\boldsymbol{q})^*\times\boldsymbol{b}(\boldsymbol{q})\right). \tag{11}$$

Expression (11) for $\boldsymbol{\sigma}(\boldsymbol{q})$ can be worked out explicitly with the help of the expressions (7) and (8) for $\boldsymbol{e}(\boldsymbol{q})$ and $\boldsymbol{b}(\boldsymbol{q})$, respectively. We split off an overall positive factor (which does not affect the field lines, since these are only determined by the direction of the vector field):

$$\boldsymbol{\sigma}(\boldsymbol{q}) = \frac{1}{q^2}e^{-2q\mathrm{Im}n}\boldsymbol{\sigma}'(\boldsymbol{q}). \tag{12}$$

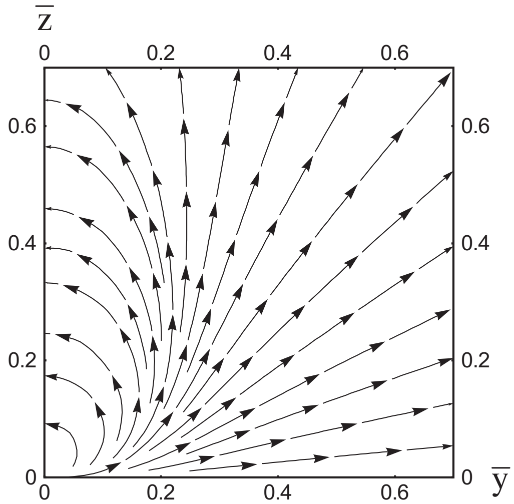

Figure 3. Shown are field lines of the Poynting vector for a dipole oscillating along the z-axis. The parameters of the medium are $\varepsilon_r = 1.7 + 0.06i$ and $\mu_r = 1$. The upward curving comes from the imaginary part of ε_r.

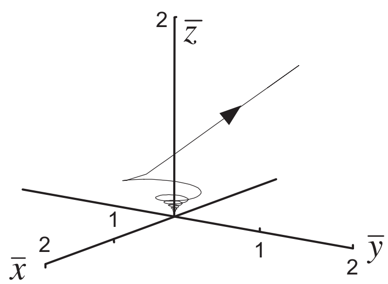

Figure 4. The graph shows one field line of the Poynting vector for a rotating dipole moment. The parameters of the medium are $\varepsilon_r = 2 + 0.1i$ and $\mu_r = 1$.

We then obtain

$$\boldsymbol{\sigma}'(\boldsymbol{q}) = [1 - (\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{u}})(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{u}}^*)]\,\hat{\boldsymbol{r}}\,\mathrm{Re}\left[\frac{n}{\mu_r}\left(1 + \frac{i}{nq}\right)\right] + \left|1 + \frac{i}{nq}\right|^2 \frac{1}{|n|^2 q}\{[1 - 3(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{u}})(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{u}}^*)]\,\hat{\boldsymbol{r}}\mathrm{Im}(\varepsilon_r) + 2\mathrm{Im}[\varepsilon_r(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{u}}^*)\hat{\boldsymbol{u}}]\}. \quad (13)$$

The vector $\boldsymbol{\sigma}'(\boldsymbol{q})$ defines a vector field, and electromagnetic energy flows along its field lines. A field line through a given point $(\bar{x}_o, \bar{y}_o, \bar{z}_o)$ is the solution of $\mathrm{d}\boldsymbol{q}/\mathrm{d}u = \boldsymbol{\sigma}'$ with $\boldsymbol{q}(u)$ the parametrization of the field line with the dummy variable u. The equation $\mathrm{d}\boldsymbol{q}/\mathrm{d}u = \boldsymbol{\sigma}'$ would need to be solved numerically, with $(\bar{x}_o, \bar{y}_o, \bar{z}_o)$ as initial values, except in simple cases where a closed form solution can be found [6].

4. General effects of the embedding medium

When the distance between the dipole and the field point is more than several wave lengths, q is large, and we have

$$\boldsymbol{\sigma}'(\boldsymbol{q}) = [1 - (\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{u}})(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{u}}^*)]\ \mathrm{Re}\left(\frac{n}{\mu_r}\right)\hat{\boldsymbol{r}} + \ldots \quad (14)$$

This is the far field, and we see that $\boldsymbol{\sigma}'(\boldsymbol{q})$ is proportional to $\hat{\boldsymbol{r}}$. Therefore, in the far field, the field lines run radially outward. Inspection of Equation (13) shows that all terms are proportional to $\hat{\boldsymbol{r}}$, except the one containing $2\mathrm{Im}[\varepsilon_r(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{u}}^*)\hat{\boldsymbol{u}}]$. Any curving of field lines comes from this term, and since it has an overall $1/q$ factor, this curving can only occur in the near field.

For a linear dipole, oscillating along the z-axis, we have $\hat{\boldsymbol{u}} = \boldsymbol{e}_z$ and $2\mathrm{Im}[\varepsilon_r(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{u}}^*)\hat{\boldsymbol{u}}]$ is equal to $2\boldsymbol{e}_z \cos\theta \mathrm{Im}\varepsilon_r$. Therefore, any curving of the field lines comes from the imaginary part of ε_r. The field lines of the Poynting vector, e.g. $\boldsymbol{\sigma}'(\boldsymbol{q})$, lie in a plane containing the z-axis. Due to $2\boldsymbol{e}_z \cos\theta \mathrm{Im}\varepsilon_r$, the vector $\boldsymbol{\sigma}'(\boldsymbol{q})$ has a part proportional to $\boldsymbol{e}_z$, in addition to the remaining part which is proportional to $\hat{\boldsymbol{r}}$. This makes the field lines bend away from the xy-plane. A typical field line diagram is shown in Figure 3. The field lines are rotationally symmetric around the z-axis, and the picture is reflection symmetric in the xy-plane. A more detailed analysis of this phenomenon can be found in Ref. [10].

For a circular dipole, vector $\hat{\boldsymbol{u}}$ is given by Equation (2). We now have

$$2\mathrm{Im}[\varepsilon_r(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{u}}^*)\hat{\boldsymbol{u}}] = (\hat{\boldsymbol{r}} - \boldsymbol{e}_z \cos\theta)\mathrm{Im}\varepsilon_r + \boldsymbol{e}_\phi \sin\theta \mathrm{Re}\varepsilon_r. \quad (15)$$

The imaginary part of ε_r adds an $\boldsymbol{e}_z$ component, just like for the linear dipole. As a result, the tight windings shown in Figure 2 become thinner, and the cone shape becomes more like a funnel shape. A typical field line is shown in Figure 4. We also notice that the spatial extent of the vortex has diminished. The real part of ε_r is proportional to $\boldsymbol{e}_\phi$. This yields the rotation around the z-axis, and so this part is responsible for the appearance of the vortex. In Figures 2 and 4, the field lines wind around the z-axis with the same orientation as the rotation of the dipole moment in the xy-plane. For materials with a negative real part of ε_r, the rotation of the field lines is opposite to the rotation of the dipole moment [8].

5. Single-negative materials

The curving of the field lines in Figure 3 and the vortex in Figure 4 are near-field phenomena. In the far field, field lines approach a straight line, since $\boldsymbol{\sigma}'(\boldsymbol{q})$, given by Equation (14) is proportional to $\hat{\boldsymbol{r}}$. The material parameters enter through $\mathrm{Re}\,(n/\mu_r)$, and it is easy to show that

$$\mathrm{Re}\left(\frac{n}{\mu_r}\right) \geq 0. \quad (16)$$

The equal sign holds if and only if both ε_r and μ_r are real, and are of opposite sign. Such materials are called single-negative materials. A simple example is a metal

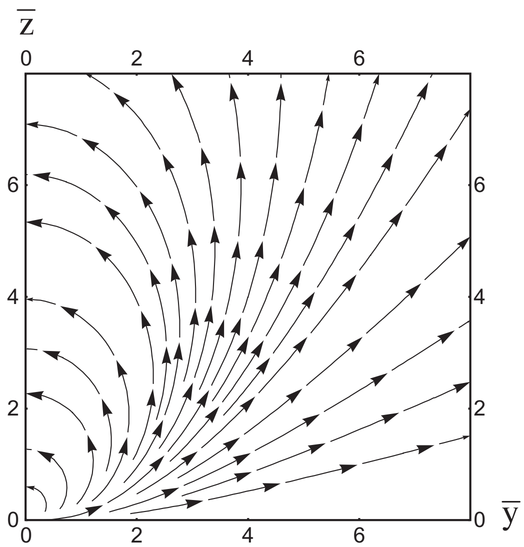

Figure 5. The figure shows the field lines of the Poynting vector for a dipole oscillating along the z-axis, and embedded in a near-single-negative material with parameters $\varepsilon_r = -1.7 + 0.06i$ and $\mu_r = 1$. The extent of the curving is about 10 times larger than in Figure 3.

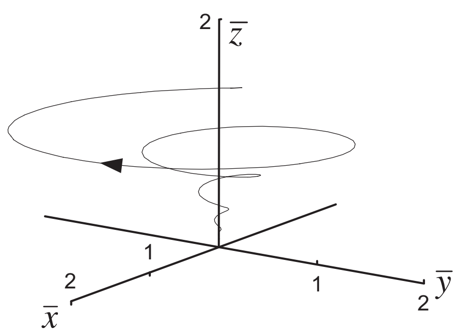

Figure 6. Shown is a field line of the Poynting vector for a rotating dipole, embedded in a medium with $\varepsilon_r = -0.8 + 0.1i$ and $\mu_r = 1$. We notice the thin windings, due to $\mathrm{Im}\varepsilon_r$, and the large spatial extent, due to the fact that the material is near-single-negative.

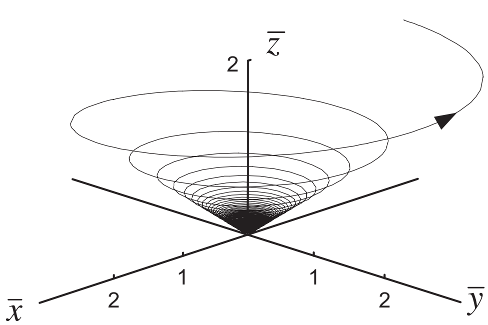

Figure 7. Shown is a field line of the Poynting vector for a rotating dipole, embedded in a medium with $\mu_r = -0.8 + 0.1i$ and $\varepsilon_r = 1$. The windings are tight, since $\mathrm{Im}\varepsilon_r = 0$, and the vortex is huge, since the medium is near-single-negative.

without damping. For these materials, the first term on the right-hand side of Equation (14) vanishes, and therefore there is no far field. Or, from a different point of view, the near field extends to infinity. Expression (13) simplifies to

$$\boldsymbol{\sigma}'(\boldsymbol{q}) = \frac{2\varepsilon_r}{|n|^2 q}\left(1 + \frac{1}{|n|q}\right)^2 \mathrm{Im}[(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{u}}^*)\hat{\boldsymbol{u}}]. \quad (17)$$

The radial component is

$$\boldsymbol{\sigma}'(\boldsymbol{q}) \cdot \hat{\boldsymbol{r}} = 0, \quad (18)$$

since $\mathrm{Im}[(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{u}}^*)(\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{r}})] = 0$. Therefore, there is no power outflow in the radial direction, and consequently, the emitted power per unit solid angle is $\mathrm{d}P/\mathrm{d}\Omega = 0$ for any direction and any distance q from the dipole. Field lines of the Poynting vector are parametrized as $\boldsymbol{q}(u)$, and the curves are the solution of $\mathrm{d}\boldsymbol{q}/\mathrm{d}u = \boldsymbol{\sigma}'(\boldsymbol{q})$. When using spherical coordinates (q, θ, ϕ) for the representation of points on the field line, we find $\mathrm{d}q/\mathrm{d}u = \boldsymbol{\sigma}'(\boldsymbol{q}) \cdot \hat{\boldsymbol{r}} = 0$ for the u dependence of q. This implies that q is constant along a field line, and therefore, any field line of the Poynting vector is a curve on a sphere around the dipole.

For a linear dipole, we take $\hat{\boldsymbol{u}} = \boldsymbol{e}_z$. Then, $\mathrm{Im}[(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{u}}^*)\hat{\boldsymbol{u}}] = 0$ and $\boldsymbol{\sigma}'(\boldsymbol{q}) = 0$ for all field points. There is no power flow anywhere. It can be shown that $\boldsymbol{e}(\boldsymbol{q})$ is real and $\boldsymbol{b}(\boldsymbol{q})$ is imaginary. This makes $\boldsymbol{e}(\boldsymbol{q})^* \times \boldsymbol{b}(\boldsymbol{q})$ imaginary, and since μ_r is real, we find $\boldsymbol{\sigma}'(\boldsymbol{q}) = 0$ with Equation (11). For a realistic material, ε_r and μ_r will contain at least a small imaginary part. Figure 5 shows the field lines of the Poynting vector for $\varepsilon_r = -1.7 + 0.06i$ and $\mu_r = 1$. The picture has a similar appearance as the field line distribution in Figure 3 for which $\varepsilon_r = +1.7 + 0.06i$ and $\mu_r = 1$. However, the scale differs by a factor of 10. The extent of the curving in the near field in Figure 5 is about 10 times larger for the case of Figure 3.

For a dipole rotating in the xy-plane, we take $\hat{\boldsymbol{u}}$ as in Equation (2). We then have $2\mathrm{Im}[(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{u}}^*)\hat{\boldsymbol{u}}] = \boldsymbol{e}_\phi \sin\theta$, and the Poynting vector becomes

$$\boldsymbol{\sigma}'(\boldsymbol{q}) = \frac{\varepsilon_r \sin\theta}{|n|^2 q}\left(1 + \frac{1}{|n|q}\right)^2 \boldsymbol{e}_\phi. \quad (19)$$

This vector is proportional to $\boldsymbol{e}_\phi$, and therefore, the field lines are circles around the z-axis. For ε_r positive, the direction of the field lines is in the same direction as the rotation of the dipole (counterclockwise when viewed down the positive z-axis). For ε_r negative, the field lines run opposite to the rotation direction of the dipole. In general, the sign of $\mathrm{Re}\varepsilon_r$ determines the rotation direction. Figure 6 shows a field line of the Poynting vector for $\varepsilon_r = -0.8 + 0.1i$ and $\mu_r = 1$. As compared to Figure 4, for which $\varepsilon_r = 2 + 0.1i$ and $\mu_r = 1$, the spatial extent of the vortex in Figure 6 is huge. For Figure 7, we have $\varepsilon_r = 1$ and

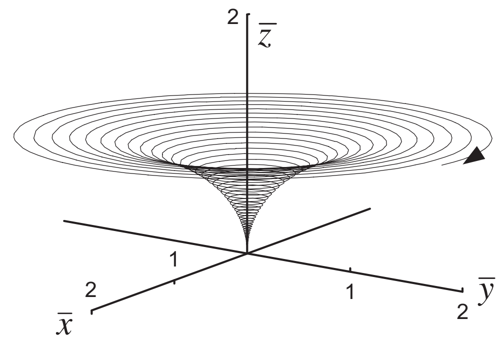

Figure 8. The figure shows a field line of energy flow for a rotating dipole and a very-near-single-negative medium. The parameters are $\varepsilon_r = -0.8 + 0.01i$ and $\mu_r = 1$.

$\mu_r = -0.8 + 0.1i$. As compared to Figure 2, for a dipole in free space, we see again that for a near-single-negative material, the dimension of the vortex becomes enormous. Usually, the value of μ_r has a negligible effect on the energy flow, but the case of a near-single-negative material is an exception. For Figure 8, we have $\varepsilon_r = -0.8 = 0.01i$ and $\mu_r = 1$. These parameters are similar to the parameters for Figure 6, except that the imaginary part of ε_r is 10 times smaller. So, the case of Figure 8 is closer to a perfect single-negative material. We see that the windings of the field line are much tighter, the funnel shape is more pronounced, and the field lines are near circular. Also the extent of the vortex is much larger than in Figure 6, and, in fact, is much larger than depicted in the figure. This is 'the giant dipole vortex', which is unique to single-negative materials.

6. Conclusions

The pattern of energy flow of radiation emitted by an electric dipole is strongly affected by the medium into which the radiation is emitted. Absorption by the material does not only weaken the energy flow, but it also redirects it. An imaginary part of ε_r makes the field lines of the Poynting vector bend towards the z-axis. For a linear dipole, the z-axis is the oscillation direction of the dipole, and for a circular dipole, the z-axis is the axis perpendicular to the plane of rotation. For a circular dipole, the field lines wind around the z-axis and the sign of the real part of ε_r determines the direction of rotation. The value of μ_r has in general very little effect on the pattern of energy flow. A typical field line distribution for a linear dipole is shown in Figure 3 and a typical field line for a circular dipole is depicted in Figure 4.

The curving of the field lines is a near-field phenomenon, and the structures shown in Figures 3 and 4 have a spatial extent of at most an optical wavelength. Further out, in the far field, the field lines approach straight lines. An exception to this occurs when the embedding material is single-negative. In this case, the dominating term in the far field (the right-hand side of Equation (14)) vanishes. For near-single-negative materials, the far-field term only becomes significant at very large distances from the dipole, and therefore, the near-field curving continues up to large distances from the dipole. The effect is most dramatic for a circular dipole, as shown in Figures 6–8. When the material is very-near-single-negative, as in Figure 8, a 'giant dipole vortex' appears.

Disclosure statement

No potential conflict of interest was reported by the authors.

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