Theory of field attenuation in photon detection, with an application to resonance fluorescence

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Detection of photons from electromagnetic radiation can be considered as the appearance of random events on the time axis. When an attenuator is placed in front of the detector, which attenuates the intensity by a factor of α, the statistical properties of the detected photons are altered. We show that simple relations exist between the statistical functions of the photons detected from the attenuated field and the same functions for the photons that would be detected from the unattenuated field. We also derive several recurrence relations for the statistical functions involving their dependence on the parameter α. For photon detection from resonance fluorescence, the parameter α appears naturally as the probability that an emitted photon is detected. In this case, there is no attenuator, but the parameter α appears in the same way. We show that the probability for the emission ($\alpha = 1$) of n photons in a given time interval can easily be computed, and with the general theory we can then obtain the result for the detection of n photons ($\alpha < 1$).

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1. INTRODUCTION

When light is detected with a photomultiplier tube, photoelectric pulses are recorded, and these events are interpreted as observations of photons. Due to the quantum mechanical nature of the interaction between the incoming light and the sensitive part of the photomultiplier, photons seem to appear randomly. Photon counts are considered to be random events (point process) on the time axis, and when the appearance of photons would be purely random, this would be a Poisson process. The landmark experiment of Hanbury Brown and Twiss [1–3] showed for the first time that photons can be correlated. This implies that the observation of one photon influences the probability for the detection of a second photon at a later time. For a Poisson process, photons appear independently, without a memory to photon detections in the past. Photons in a laser beam are independent, and the probability distribution is a Poisson distribution. Photons in thermal light, or any other source with a classical description, are bunched. This means that the observation of the first photon enhances the probability for the detection of a second photon immediately afterward [4]. Fluorescent photons emitted by a two-state atom in a laser beam are antibunched [5,6], and the probability for the detection of the second photon immediately after the first is zero. The corresponding statistics is sub-Poissonian, with the variance in the photon count smaller than the average [7,8].

The statistical properties of random events are most conveniently represented by the intensity correlation functions, defined as [9]

$$I_k(t_1, \ldots, t_k)dt_1 \ldots dt_k$$
$$= \text{probability for an event in } [t_1, t_1 + dt_1], \text{ and} \ldots \text{and an event in } [t_k, t_k + dt_k], \text{ irrespective of events at other times, and with } t_1 < \cdots < t_k. \quad (1)$$

The function with $k = 1$ will usually be written as I, rather than I_1, and $I(t)$ is called the intensity of the process. For independent events (Poisson process), the correlation functions factor as

$$I_k(t_1, \ldots, t_k) = I(t_1) \ldots I(t_k), \quad (2)$$

indicating that the occurrence of an event at, say, t_1 does not influence the probabilities for the occurrence of events at other times. For this case, all statistical properties of the random process can be expressed in terms of the intensity $I(t)$.

When light is detected with a photomultiplier, the incident electric field $\mathbf{E}(\mathbf{r}, t)$ determines the response of the detector. We shall assume that the field is polarized, and indicate by $E(t)$ the projection of $\mathbf{E}(\mathbf{r}, t)$ onto the polarization direction, and evaluation at the position of the detector. It can then be shown that the photon intensity correlation functions are given by [10,11]

$$I_k(t_1, \ldots, t_k) = \zeta^k \langle E(t_1)^{(-)} \ldots E(t_k)^{(-)} E(t_k)^{(+)} \ldots E(t_1)^{(+)} \rangle \quad (3)$$

and here $(+)$ and $(-)$ indicate the positive and negative frequency parts of $E(t)$, respectively. Parameter ζ is an overall constant, and $\langle \ldots \rangle$ indicates an average. For an incident quantum field, this is a quantum expectation value, and for an incident classical field this represents an average over possible stochastic fluctuations in the radiation. When the incident field is a classical, deterministic field, the right-hand side of Eq. (3) becomes $I(t_1)\ldots I(t_k)$, with $I(t) = \zeta E(t)^{(-)}E(t)^{(+)}$, and therefore the photon detections are independent. For quantum radiation, the $2k$ functions inside $\langle\ldots\rangle$ are operators, which will in general not commute. This gives rise to correlations between the photons. Similarly, for a randomly fluctuating classical field, the right-hand side of Eq. (3) will not factor as in Eq. (2), and therefore the photon detections will be correlated.

2. RANDOM EVENTS

The number of events in a time interval $[t_a, t_b]$ is a random number, and we indicate by $P_n(t_a, t_b)$ the probability that n events occur in $[t_a, t_b]$. The factorial moments of the process are defined as

$$S_k(t_a, t_b) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} P_n(t_a, t_b), \qquad k = 0, 1, 2, \ldots, \tag{4}$$

and the generating function is defined as

$$G(x, t_a, t_b) = \sum_{n=0}^{\infty} x^n P_n(t_a, t_b). \tag{5}$$

When we expand $G(x, t_a, t_b)$ in a Taylor series around $x = 1$ we find

$$G(x, t_a, t_b) = \sum_{k=0}^{\infty} \frac{(x-1)^k}{k!} S_k(t_a, t_b). \tag{6}$$

The right-hand side of Eq. (5) is a Taylor series around $x = 0$, and therefore

$$P_n(t_a, t_b) = \frac{1}{n!} \frac{\partial^n}{\partial x^n} G(x, t_a, t_b)|_{x=0}. \tag{7}$$

Then we substitute $G(x, t_a, t_b)$ from Eq. (6), and this yields

$$P_n(t_a, t_b) = \frac{(-1)^n}{n!} \sum_{k=n}^{\infty} \frac{(-1)^k}{(k-n)!} S_k(t_a, t_b), \tag{8}$$

which is the inverse of the relation in Eq. (4).

The factorial moments can be found from the intensity correlations in Eq. (1) as [12]

$$S_0(t_a, t_b) = 1, \tag{9}$$

$$S_1(t_a, t_b) = \int_{t_a}^{t_b} \mathrm{d}t I(t), \tag{10}$$

$$S_k(t_a, t_b) = k! \int_{t_a}^{t_b} \mathrm{d}t_k \int_{t_a}^{t_k} \mathrm{d}t_{k-1} \ldots \int_{t_a}^{t_2} \mathrm{d}t_1 I_k(t_1, \ldots, t_k), \quad k = 2, 3, \ldots. \tag{11}$$

Once the factorial moments are known, the probabilities $P_n(t_a, t_b)$ can be obtained from Eq. (8).

The average number of events, $\mu(t_a, t_b)$, in $[t_a, t_b]$ is

$$\mu(t_a, t_b) = \sum_{n=0}^{\infty} n P_n(t_a, t_b) = S_1(t_a, t_b) \tag{12}$$

and with Eq. (10) we then find

$$I(t) = \frac{\partial}{\partial t} \mu(t_a, t). \tag{13}$$

Therefore, the intensity of the process can be found from the time dependence of the probabilities $P_n(t_a, t_b)$.

Let τ_n be the time at which the nth event occurs, after the initial time t_a. So, τ_n is the waiting time for the nth event. This τ_n is a random variable, which has a probability density function $w_n(t_a, t)$. Therefore, we define

$$\begin{aligned} w_n(t_a, t)dt &= \text{probability that the}\, n\text{th event occurs in}\, [t, t+dt], \\ &= \text{probability for}\, n-1\, \text{events in}\, [t_a, t]\, \text{and an event} \\ &\quad \text{in}\, [t, t+dt]. \end{aligned} \tag{14}$$

It can be shown [13] that these probability densities can be found from the probabilities according to

$$w_n(t_a, t) = -\frac{\partial}{\partial t} \sum_{m=0}^{n-1} P_m(t_a, t). \tag{15}$$

Also of interest is the conditional probability for n events in an observation time interval, after an event at the initial time t_a. This conditional probability is defined as

$$\begin{aligned} P_n(t_a, t_b|t_a) &= \text{probability for}\, n\, \text{events in}\, [t_a, t_b],\, \text{after an} \\ &\quad \text{event in}\, [t_a - dt_a, t_a]. \end{aligned} \tag{16}$$

It can then be shown that [13]

$$P_n(t_a, t_b|t_a) = \frac{1}{I(t_a)} \frac{\partial}{\partial t_a} \sum_{m=0}^{n} P_m(t_a, t_b), \tag{17}$$

where $I(t_a)dt_a$ equals the probability for an event in $[t_a - dt_a, t_a]$. Similarly, the conditional probability densities are defined as

$$\begin{aligned} w_n(t_a, t|t_a)dt &= \text{probability that the}\, n\text{th even to occurs in} \\ &\quad [t, t+dt], \text{after an event in}\, [t_a - dt_a, t_a]. \end{aligned} \tag{18}$$

We find that [13]

$$w_n(t_a, t|t_a) = \frac{1}{I(t_a)} \frac{\partial}{\partial t_a} \sum_{m=1}^{n} w_m(t_a, t). \tag{19}$$

3. SUM RULES

When we sum Eq. (8) over n, and change the order of summation, we find

$$\sum_{n=0}^{\infty} P_n(t_a, t_b) = S_0(t_a, t_b) = 1. \tag{20}$$

The left-hand side equals the probability to find any number of events in $[t_a, t_b]$, so this must obviously equals unity. From Eq. (8), however, it follows that the sum equals $S_0(t_a, t_b)$, and with Eq. (9) we have $S_0(t_a, t_b) = 1$. Let us assume that the intensity correlations $I_k(t_1, \ldots, t_k)$ are known. Then the factorial moments $S_k(t_a, t_b)$ follow from Eqs. (9)–(11), and subsequently the probabilities $P_n(t_a, t_b)$ can be obtained from Eq. (8). We then see that the sum of the P_n's equals unity because we set $S_0(t_a, t_b) = 1$ in Eq. (9). Therefore, this sum rule is automatically satisfied for any set of intensity correlation functions $I_k(t_1, \ldots, t_k)$.

When we set $x = 0$ in Eqs. (5) and (6) we find

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} S_k(t_a, t_b) = P_0(t_a, t_b). \tag{21}$$

This sum rule for the factorial moments is similar in form to Eq. (20).

With Eq. (20), Eq. (15) can also be written as

$$w_n(t_a, t) = \frac{\partial}{\partial t} \sum_{m=n}^{\infty} P_m(t_a, t). \tag{22}$$

Summing over n and changing the order of summation yields

$$\sum_{n=1}^{\infty} w_n(t_a, t) = \frac{\partial}{\partial t} \sum_{m=0}^{\infty} m P_m(t_a, t). \tag{23}$$

With Eq. (13), this is

$$\sum_{n=1}^{\infty} w_n(t_a, t) = I(t), \tag{24}$$

which is the sum rule for $w_n(t_a, t)$. When we multiply the left-hand side by dt, this equals the probability to find any number of events in $[t_a, t]$ and an event in $[t, t + dt]$, and according to Eq. (1) this is $I(t)dt$.

The sum rule for the conditional probabilities $P_n(t_a, t_b|t_a)$ can be obtained along similar lines, and we find

$$\sum_{n=0}^{\infty} P_n(t_a, t_b|t_a) = 1. \tag{25}$$

This should be so, because the conditional probabilities are probabilities. In order to obtain the sum rule for the conditional probability densities $w_n(t_a, t|t_a)$, we note that $w_n(t_a, t|t_a) I(t_a) dt_a dt$ equals the probability for an event in $[t_a - dt_a, t_a]$, an event in $[t, t + dt]$, and $n - 1$ events in $[t_a, t_b]$. When we sum this over n, this yields $I_2(t_a, t) dt_a dt$, and therefore

$$\sum_{n=1}^{\infty} w_n(t_a, t|t_a) = \frac{I_2(t_a, t)}{I(t_a)}. \tag{26}$$

4. PHOTON COUNTING AND ATTENUATION

When light is detected with a photomultiplier tube, photons appear as random events on the time axis. The incident electric field determines the intensity correlation functions, according to Eq. (3). Once these functions are known, the probabilities and probability densities, both conditional and unconditional, can then be obtained, as outlined in Section 2. Now let us assume that an attenuator is placed in front of the photomultiplier. This could be, for instance, an absorbing slab of dielectric material. The attenuated field then has a positive frequency part $\beta E(t)^{(+)}$, with β a complex number, and $|\beta| \le 1$. The negative frequency part of the electric field picks up a factor β^*, and therefore the intensity correlation functions acquire an overall factor of α^k, with $\alpha = |\beta|^2$. We shall compare the counting statistics of the attenuated field to the counting statistics of the unattenuated field ($\alpha = 1$), and show that these are related in a rather simple way. From here on we shall display explicitly the α dependence of the various statistical quantities. So, for instance, $P_n(t_a, t_b; \alpha)$ is the probability to observe n photons in $[t_a, t_b]$, when the field is attenuated by a factor α. For the intensity correlations we have the obvious relation

$$I_k(t_1, \ldots, t_k; \alpha) = \alpha^k I_k(t_1, \ldots, t_k; 1). \tag{27}$$

For the intensity of the process we have $I(t; \alpha) = \alpha I(t; 1)$. If the process with $\alpha = 1$ is a Poisson process, then the process with $\alpha \ne 1$ is also a Poisson process, as follows immediately from Eq. (2).

With Eqs. (9)–(11) we find

$$S_k(t_a, t_b; \alpha) = \alpha^k S_k(t_a, t_b; 1). \tag{28}$$

The α dependence of the generating function follows from Eqs. (6) and (28), which gives

$$G(x, t_a, t_b; \alpha) = G(\alpha(x - 1) + 1, t_a, t_b; 1). \tag{29}$$

So, if the x dependence of $G(x, t_a, t_b; 1)$ is known, we replace x by $\alpha(x - 1) + 1$ in this function, which then yields $G(x, t_a, t_b; \alpha)$.

The probabilities $P_n(t_a, t_b; \alpha)$ are given by Eq. (8), where we use Eq. (28) for the factorial moments. Then, for $S_k(t_a, t_b; 1)$ we substitute Eq. (1), which leads to a double sum. When we change the order of summation, then the inner sum has the form of Newton's binomium. We thus obtain

$$P_n(t_a, t_b; \alpha) = \sum_{m=n}^{\infty} \binom{m}{n} \alpha^n (1 - \alpha)^{m-n} P_m(t_a, t_b; 1). \tag{30}$$

This expression is sometimes referred to as a Bernoulli convolution. We can interpret this result as follows. Let α be the probability that a photon which is incident on the attenuator will be transmitted. Then $1 - \alpha$ is the probability that an incident photon is not transmitted. So $\alpha^n (1 - \alpha)^{m-n}$ is the probability that a particular set of n photons will be transmitted, if there are m incident photons. The binomial coefficient equals the number of ways we can pick n out of m. Such an argument relies on the interpretation that the incident field can be viewed as a stream of photons, and that these photons are independent. Apparently, Eq. (30) holds in general for the statistics of photon detection.

When we sum both sides of Eq. (30) over n, we get a double sum on the right-hand side. Changing the order of summation yields

$$\sum_{n=0}^{\infty} P_n(t_a, t_b; \alpha) = \sum_{m=0}^{\infty} P_m(t_a, t_b; 1). \qquad (31)$$

Therefore, the sum rule (20) is preserved upon attenuation.

5. PROBABILITY DENSITIES OF THE ATTENUATED FIELD

The probability densities are determined by the probabilities as in Eq. (22):

$$w_n(t_a, t; \alpha) = \frac{\partial}{\partial t} \sum_{m=n}^{\infty} P_m(t_a, t; \alpha). \qquad (32)$$

Here, the α dependence is shown explicitly. For $P_m(t_a, t; \alpha)$, we substitute the right-hand side of Eq. (30), and change the order of summation. This yields

$$w_n(t_a, t; \alpha) = \frac{\partial}{\partial t} \sum_{m=n}^{\infty} a_{n,m}(\alpha) P_m(t_a, t; 1). \qquad (33)$$

The combinatorial functions $a_{n,m}(\alpha)$ are defined as

$$a_{n,m}(\alpha) = \sum_{k=n}^{m} \binom{m}{k} \alpha^k (1-\alpha)^{m-k}, \qquad m \geq n \geq 0. \qquad (34)$$

Comparison of Eqs. (32) and (33) shows that the α dependence is effectively factored out of $P_m(t_a, t; \alpha)$ as $a_{n,m}(\alpha)$. For $\alpha = 1$ we have

$$a_{n,m}(1) = 1. \qquad (35)$$

In order to express $w_n(t_a, t; \alpha)$ in terms of $w_n(t_a, t; 1)$, we use

$$\frac{\partial}{\partial t} P_m(t_a, t; 1) = w_m(t_a, t; 1) - w_{m+1}(t_a, t; 1), \qquad m \geq 1, \quad (36)$$

as can be verified from Eq. (15). Then Eq. (33) becomes

$$w_n(t_a, t; \alpha) = \alpha^n w_n(t_a, t; 1) + \sum_{m=n+1}^{\infty} [a_{n,m}(\alpha) - a_{n,m-1}(\alpha)] w_m(t_a, t; 1), \quad (37)$$

and here we have used

$$a_{n,n}(\alpha) = \alpha^n. \qquad (38)$$

The right-hand side of Eq. (37) can be simplified. To this end, we introduce a generating function in m for $a_{n,m}(\alpha)$ as

$$g_n(y; \alpha) = \sum_{m=n}^{\infty} a_{n,m}(\alpha) y^m, \qquad n \geq 0. \qquad (39)$$

We substitute the right-hand side of Eq. (34) for $a_{n,m}(\alpha)$, and change the order of summation. The inner sum is of the form

$$\sum_{m=k}^{\infty} \frac{m!}{(m-k)!} u^m = \frac{k! u^k}{(1-u)^{k+1}}, \qquad (40)$$

and after this, the remaining series is a geometric series. We thus find

$$g_n(y; \alpha) = \frac{1}{1-y} \left(\frac{\alpha y}{1 - y + \alpha y} \right)^n. \qquad (41)$$

Then we differentiate both sides with respect to y, and compare the coefficients of y^m on both sides. This yields the recurrence relation

$$a_{n,n+1}(\alpha) = [1 + (1-\alpha)n] a_{n,n}(\alpha), \qquad (42)$$

$$(m-n) a_{n,m}(\alpha) + [(\alpha - 1)(m-1) + n - m] a_{n,m-1}(\alpha) + (1-\alpha)(m-1) a_{n,m-2} = 0, \quad m \geq n+2. \qquad (43)$$

This is a three-term recurrence relation in m, for n fixed, and the initial value is given by Eq. (38). We can also write Eq. (43) as

$$(m-n)[a_{n,m}(\alpha) - a_{n,m-1}(\alpha)] = (1-\alpha)(m-1)[a_{n,m-1}(\alpha) - a_{n,m-2}(\alpha)], \quad m \geq n+2, \quad (44)$$

and this is a two-term recurrence relation for $a_{n,m}(\alpha) - a_{n,m-1}(\alpha)$. Solving by iteration gives

$$a_{n,m}(\alpha) - a_{n,m-1}(\alpha) = (1-\alpha)^{m-n-1} \frac{(m-1)!}{n!(m-n)!} [a_{n,n+1}(\alpha) - a_{n,n}(\alpha)], \qquad (45)$$

and with Eqs. (38) and (42) we find

$$a_{n,n+1}(\alpha) - a_{n,n}(\alpha) = n(1-\alpha)\alpha^n. \qquad (46)$$

This finally gives

$$a_{n,m}(\alpha) - a_{n,m-1}(\alpha) = \binom{m-1}{n-1} \alpha^n (1-\alpha)^{m-n}, \qquad m \geq n+1. \qquad (47)$$

With Eq. (47), Eq. (37) becomes

$$w_n(t_a, t; \alpha) = \sum_{m=n}^{\infty} \binom{m-1}{n-1} \alpha^n (1-\alpha)^{m-n} w_m(t_a, t; 1), \qquad (48)$$

which is the desired expression for $w_n(t_a, t; \alpha)$ in terms of $w_n(t_a, t; 1)$. When we sum both sides over n, we obtain the relation

$$\sum_{n=1}^{\infty} w_n(t_a, t; \alpha) = \alpha \sum_{m=1}^{\infty} w_m(t_a, t; 1). \qquad (49)$$

According to Eq. (24), the left-hand side is $I(t; \alpha)$ and the right-hand side is $\alpha I(t; 1)$. So Eq. (49) expresses the relation

$$I(t; \alpha) = \alpha I(t; 1), \qquad (50)$$

which is Eq. (27) with $k = 1$.

6. CONDITIONAL PROBABILITIES AND PROBABILITY DENSITIES OF THE ATTENUATED FIELD

The conditional probabilities for photon detection from the attenuated field can be obtained as follows. We write Eq. (17) as

$$P_n(t_a,t_b|t_a;\alpha) = -\frac{1}{I(t_a;\alpha)}\frac{\partial}{\partial t_a}\sum_{m=n+1}^{\infty} P_m(t_a,t_b;\alpha), \tag{51}$$

where we have used that the P_m's sum to unity. With Eq. (30) we have

$$P_m(t_a,t_b;\alpha) = \sum_{k=m}^{\infty}\binom{k}{m}\alpha^m(1-\alpha)^{k-m}P_k(t_a,t_b;1). \tag{52}$$

Then we substitute this in the right-hand side of Eq. (51), and change the order of summation. This yields

$$P_n(t_a,t_b|t_a;\alpha) = -\frac{1}{\alpha I(t_a;1)}\frac{\partial}{\partial t_a}\sum_{m=n+1}^{\infty} a_{n+1,m}(\alpha)P_m(t_a,t_b;1), \tag{53}$$

where we have used Eq. (50) and the definition in Eq. (34) of $a_{n,m}(\alpha)$. It can be shown from Eq. (17) that

$$\frac{1}{I(t_a;1)}\frac{\partial}{\partial t_a}P_m(t_a,t_b;1) = P_m(t_a,t_b|t_a;1) - P_{m-1}(t_a,t_b|t_a;1),\quad m\geq 1, \tag{54}$$

and with this, Eq. (53) becomes

$$P_n(t_a,t_b|t_a;\alpha) = \alpha^n P_n(t_a,t_b|t_a;1) + \frac{1}{\alpha}\sum_{m=n+1}^{\infty}[a_{n+1,m+1}(\alpha) - a_{n+1,m}(\alpha)]P_m(t_a,t_b|t_a;1). \tag{55}$$

Here we have used Eq. (38). With Eq. (47) we then obtain the final result

$$P_n(t_a,t_b|t_a;\alpha) = \sum_{m=n}^{\infty}\binom{m}{n}\alpha^n(1-\alpha)^{m-n}P_n(t_a,t_b|t_a;1). \tag{56}$$

This expression is identical in form to the relation for the probabilities, Eq. (30).

The conditional probability densities are given by Eq. (19). With the sum rule (24), this can also be written as

$$w_n(t_a,t|t_a;\alpha) = \frac{1}{I(t_a;\alpha)}\frac{\partial}{\partial t_a}\left[I(t;\alpha) - \sum_{m=n+1}^{\infty} w_m(t_a,t;\alpha)\right], \tag{57}$$

and with Eq. (50) this is

$$w_n(t_a,t|t_a;\alpha) = -\frac{1}{\alpha I(t_a;1)}\frac{\partial}{\partial t_a}\sum_{m=n+1}^{\infty} w_m(t_a,t;\alpha). \tag{58}$$

Then we use Eq. (48) for $w_m(t_a,t;\alpha)$ and change the order of summation. We then obtain

$$w_n(t_a,t|t_a;\alpha) = -\frac{1}{I(t_a;1)}\frac{\partial}{\partial t_a}\sum_{m=n+1}^{\infty} a_{n,m-1}(\alpha)w_m(t_a,t;\alpha). \tag{59}$$

From Eq. (19) we derive

$$\frac{1}{I(t_a;1)}\frac{\partial}{\partial t_a}w_m(t_a,t;1) = w_m(t_a,t|t_a;1) - w_{m-1}(t_a,t|t_a;1),\quad m\geq 2. \tag{60}$$

Along the same lines as in the previous section we then find

$$w_n(t_a,t|t_a;\alpha) = \sum_{m=n}^{\infty}\binom{m-1}{n-1}\alpha^n(1-\alpha)^{m-n}w_m(t_a,t|t_a;1) \tag{61}$$

for $w_n(t_a,t|t_a;\alpha)$ in terms of $w_m(t_a,t|t_a;1)$. The result is identical in form to Eq. (48) for the unconditional probability densities.

7. RECURRENCE RELATIONS

Equation (30) with $n=0$ reads

$$P_0(t_a,t_b;\alpha) = \sum_{m=0}^{\infty}(1-\alpha)^m P_m(t_a,t_b;1). \tag{62}$$

Differentiating n times with respect to α yields

$$\frac{\partial^n}{\partial\alpha^n}P_0(t_a,t_b;\alpha) = (-1)^n\sum_{m=n}^{\infty}\frac{m!}{(m-n)!}(1-\alpha)^{m-n}P_m(t_a,t_b;1), \tag{63}$$

and with Eq. (30) this gives

$$P_n(t_a,t_b;\alpha) = \frac{(-\alpha)^n}{n!}\frac{\partial^n}{\partial\alpha^n}P_0(t_a,t_b;\alpha). \tag{64}$$

Therefore, the probability to detect n photons in $[t_a,t_b]$ can be found from the α dependence of the probability to detect zero photons in $[t_a,t_b]$. This relation has been found before in [14]. We can also write Eq. (64) as

$$P_{n+1}(t_a,t_b;\alpha) = -\frac{\alpha^{n+1}}{n+1}\frac{\partial}{\partial\alpha}\left(\frac{P_n(t_a,t_b;\alpha)}{\alpha^n}\right) \tag{65}$$

and this is also

$$(n+1)P_{n+1}(t_a,t_b;\alpha) = nP_n(t_a,t_b;\alpha) - \alpha\frac{\partial}{\partial\alpha}P_n(t_a,t_b;\alpha). \tag{66}$$

Apparently, the probabilities $P_n(t_a,t_b;\alpha)$ satisfy a three-term recurrence relation. Equation (56) for the conditional probabilities has the same form as Eq. (30), and therefore the relations in Eqs. (62)–(66) also hold for $P_n(t_a,t_b|t_a;\alpha)$.

In the same way, we derive from Eq. (48)

$$\frac{1}{\alpha}w_n(t_a,t;\alpha) = \frac{(-\alpha)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial\alpha^{n-1}}\left(\frac{1}{\alpha}w_1(t_a,t;\alpha)\right). \tag{67}$$

This can also be written as a recurrence relation

$$w_{n+1}(t_a,t;\alpha) = -\frac{\alpha^{n+1}}{n}\frac{\partial}{\partial\alpha}\left(\frac{w_n(t_a,t;\alpha)}{\alpha^n}\right), \tag{68}$$

or

$$n w_{n+1}(t_a, t; \alpha) = n w_n(t_a, t; \alpha) - \alpha \frac{\partial}{\partial \alpha} w_n(t_a, t; \alpha), \tag{69}$$

and the same relations hold for the conditional probability densities $w_n(t_a, t|t_a; \alpha)$.

8. PHOTON DETECTION OF RESONANCE FLUORESCENCE

An interesting example is photon detection from resonance fluorescence radiation. When a two-state atom is irradiated by a laser beam on resonance with the electronic transition, photons will be absorbed from and emitted into the laser field (stimulated transitions), and fluorescent photons will be emitted in all directions as electric dipole radiation (spontaneous transitions). These fluorescent photons can be observed by a detector, placed outside the laser beam. Assuming that the atom is in the steady state, the intensity I of the photon detection random event process is

$$I(\alpha) = \alpha A n_e, \tag{70}$$

which is independent of time. Here, A is the Einstein coefficient for spontaneous decay, and n_e is the steady-state population of the excited state. The constant α depends on the properties of the detector, its location in the field, and its aperture. However, the number of emitted photons per unit of time by the atom is An_e, as can be shown from energy considerations, and therefore parameter α can be interpreted as the probability that an emitted photon is detected. The intensity correlation functions for $k = 2, 3, \ldots$ are [15,16]

$$I_k(t_1, \ldots, t_k; \alpha) = (\alpha A)^k f(t_k - t_{k-1}) \ldots f(t_2 - t_1) n_e. \tag{71}$$

The function $f(t)$ equals the population of the excited state at time t, under the condition that the atom is in the ground state at time zero. Obviously,

$$f(0) = 0, \tag{72}$$

and therefore $I_k(t_1, \ldots, t_k; \alpha) = 0$ when two consecutive times are equal. This is the celebrated antibunching in fluorescence. For $t \to \infty$ we have $f(\infty) = n_e$. We notice that the α dependence of the intensity correlation functions is an overall factor of α^k, just as in Eq. (27) for the intensity correlations of the attenuated field. Here, this parameter α appears as the relation between the statistics of the emitted photons and the statistics of the detected photons. In a typical experiment, this parameter can be as small as $\alpha \sim 10^{-3}$.

With Ω the Rabi frequency, the population of the excited state is

$$n_e = \frac{\hat{\Omega}^2}{1 + 2\hat{\Omega}^2}, \tag{73}$$

and here we have set $\hat{\Omega} = \Omega/A$ for the Rabi frequency in units of A. The function $f(t)$ is found to be [17]

$$f(t) = n_e \left\{ 1 - e^{-\frac{3}{4}\hat{t}} \left[\frac{3}{4\rho} \sinh(\rho \hat{t}) + \cosh(\rho \hat{t}) \right] \right\}, \tag{74}$$

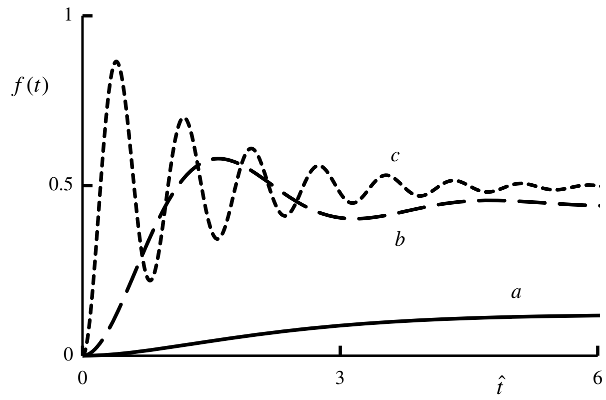

Fig. 1. Function $f(t)$ for various values of the parameter $\hat{\Omega}$. Curves a, b, and c correspond to $\hat{\Omega} = 0.4$, 2, and 8, respectively.

with $\hat{t} = At$, and

$$\rho = \sqrt{\frac{1}{16} - \hat{\Omega}^2}. \tag{75}$$

For $\hat{\Omega} < 1/4$, the function $f(t)$ has an exponential behavior. For $\hat{\Omega} > 1/4$, the parameter ρ is positive imaginary, and $f(t)$ is oscillatory. Figure 1 shows some examples of $f(t)$.

9. PHOTON PROBABILITIES OF RESONANCE FLUORESCENCE

For the atom in the steady state, the photon detection random process is stationary, and the probabilities $P_n(t_a, t_b; \alpha)$ only depend on t_a and t_b through $T = t_b - t_a$, so we consider $P_n(0, T; \alpha)$. The intensity correlations [Eq. (71)] determine the factorial moments with Eqs. (9)–(11). The multiple integrals are most conveniently dealt with by adopting a Laplace transform in T. We set

$$\tilde{S}_k(0, s; \alpha) = \int_0^\infty e^{-sT} S_k(0, T; \alpha) dT, \tag{76}$$

and similarly for other time dependent functions. Then we obtain

$$\tilde{S}_0(0, s; \alpha) = \frac{1}{s}, \tag{77}$$

$$\tilde{S}_k(0, s; \alpha) = I(\alpha) \frac{k!}{s^2} [\alpha A \tilde{f}(s)]^{k-1}, \qquad k = 1, 2, \ldots, \tag{78}$$

in terms of the Laplace transform $\tilde{f}(s)$ of $f(t)$. With Eq. (8), we find the Laplace transforms of the probabilities:

$$\tilde{P}_0(0, s; \alpha) = \frac{1}{s} - \frac{I(\alpha)}{s^2} \frac{1}{1 + \alpha A \tilde{f}(s)}, \tag{79}$$

$$\tilde{P}_n(0, s; \alpha) = \frac{I(\alpha)}{s^2} \frac{[\alpha A \tilde{f}(s)]^{n-1}}{[1 + \alpha A \tilde{f}(s)]^{n+1}}, \qquad n = 1, 2, \ldots. \tag{80}$$

We notice that the α dependence in Eqs. (79) and (80) is nontrivial. However, it can be checked by inspection that $P_n(0, T; 1)$ and $P_n(0, T; \alpha)$ are related as in Eq. (30).

From Eq. (74) we find

$$\tilde{f}(s) = \frac{1}{2s}\frac{\Omega^2}{\left(s+A\right)\left(s+\frac{1}{2}A\right)+\Omega^2}, \tag{81}$$

and this gives

$$\tilde{P}_0(0,s;\alpha) = \frac{1}{s} - \frac{\alpha A n_e}{s}\frac{(s+A)\left(s+\frac{1}{2}A\right)+\Omega^2}{s(s+A)\left(s+\frac{1}{2}A\right)+\Omega^2\left(s+\frac{1}{2}\alpha A\right)}. \tag{82}$$

In order to find the Laplace inverse, we need to know the poles of this function in the complex s plane, so we need to factor the denominator. This is a third degree polynomial in s. For $\alpha = 1$, however, a factor of $s + A/2$ can be split off, and the remaining polynomial is of second degree. This gives

$$\tilde{P}_0(0,s;1) = \frac{1}{s} - \frac{A n_e}{s}\frac{1}{s+\frac{1}{2}A}\frac{(s+A)\left(s+\frac{1}{2}A\right)+\Omega^2}{\left(s+\frac{1}{2}A\right)^2-(\gamma A)^2}, \tag{83}$$

where we have set

$$\gamma = \sqrt{\frac{1}{4}-\hat{\Omega}^2}. \tag{84}$$

From Eq. (80) we find

$$\tilde{P}_n(0,s;1) = A n_e\left(\frac{1}{2}A\Omega^2\right)^{n-1}\frac{\left[(s+A)\left(s+\frac{1}{2}A\right)+\Omega^2\right]^2}{\left(s+\frac{1}{2}A\right)^{n+1}\left[\left(s+\frac{1}{2}A\right)^2-(\gamma A)^2\right]^{n+1}}, \quad n = 1,2,\ldots. \tag{85}$$

Therefore, for the computation of $P_n(0,T;\alpha)$ is facilitated by computing $P_n(0,T;1)$ first. Then, the α dependence of $P_n(0,T;\alpha)$ follows from Eq. (30).

10. EVALUATION OF THE PROBABILITIES FOR $\alpha = 1$

The Laplace inverse of Eq. (83) is most easily obtained by using the Bromwich inversion integral [18]. We find

$$P_0(0,T;1) = \frac{1}{8\gamma^2}\frac{1}{1+2\hat{\Omega}^2}e^{-\frac{1}{2}\hat{T}}\left[-16\hat{\Omega}^4 + 4\gamma\sinh(\gamma\hat{T}) + (1+4\gamma^2)\cosh(\gamma\hat{T})\right], \tag{86}$$

with $\hat{T} = AT$. This result has been obtained before [13] in a different way. In order to obtain the inverse of Eq. (85), we first use the attenuation theorem:

$$P_n(0,T;1) = A n_e\left(\frac{1}{2}A\Omega^2\right)^{n-1}e^{-\frac{1}{2}\hat{T}} - \mathscr{L}^{-1}\left\{\frac{\left[s\left(s+\frac{1}{2}A\right)+\Omega^2\right]^2}{s^{n+1}[s^2-(\gamma A)^2]^{n+1}}\right\}. \tag{87}$$

The remaining inverse is computed with the Bromwich integral, and this yields

$$P_n(0,T;1) = \frac{\hat{\Omega}^{2n}}{1+2\hat{\Omega}^2}\frac{8}{(2\gamma)^{3n+2}}e^{-\frac{1}{2}\hat{T}}\sum_{k=0}^{n}\sum_{r=0}^{n-k}\frac{(\gamma\hat{T})^{n-k-r}}{(n-k-r)!} \times\left\{\frac{1}{2}[(-1)^k z_r(\gamma)e^{\gamma\hat{T}} + z_r(-\gamma)e^{-\gamma\hat{T}}]A_{n,k} - z_r(0)B_{n,k}\right\}, \quad n = 1,2,\ldots \tag{88}$$

with

$$z_0(x) = \left[x\left(x+\frac{1}{2}\right)+\hat{\Omega}^2\right]^2, \tag{89}$$

$$z_1(x) = \gamma(4x+1)\left[x\left(x+\frac{1}{2}\right)+\hat{\Omega}^2\right], \tag{90}$$

$$z_2(x) = \gamma^2\left[6x^2+3x+\frac{1}{4}+2\hat{\Omega}^2\right], \tag{91}$$

$$z_3(x) = \gamma^3(4x+1), \tag{92}$$

$$z_4(x) = \gamma^4, \tag{93}$$

$$z_r(x) = 0, \qquad r > 4. \tag{94}$$

The coefficients $A_{n,k}$ and $B_{n,k}$ are defined as

$$A_{n,k} = \sum_{m=0}^{k}\binom{n+m}{n}\binom{n+k-m}{n}2^{n-m}, \qquad n,k = 0,1,2,\ldots, \tag{95}$$

$$B_{n,k} = 0, \qquad k = 1,3,\ldots, \tag{96}$$

$$B_{n,2m} = (-1)^n 2^{2n}\binom{n+m}{n}, \qquad m = 0,1,\ldots. \tag{97}$$

The same coefficients appear in the expression for the conditional probability densities $w_n(0,t|0)$, which were obtained elsewhere [19]. Tables 1 and 2 list several values of $A_{n,k}$ and $B_{n,k}$.

Table 1. Several Values of A_{nk}

n/k	0	1	2	3
0	1	3/2	7/4	15/8
1	2	6	23/2	18
2	4	18	48	99
3	8	48	164	420

Table 2. Several Values of B_{nk}

n/k	0	1	2	3
0	1	0	1	0
1	−4	0	−8	0
2	16	0	48	0
3	−64	0	−256	0

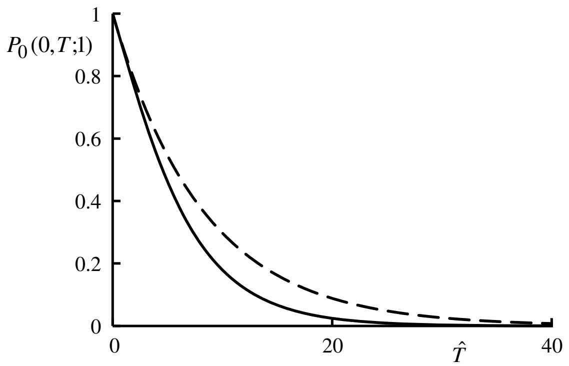

Fig. 2. Solid curve is the probability for the detection of zero photons in $[0, T]$ from resonance fluorescence radiation, for $\hat{\Omega} = 0.4$ and $\alpha = 1$. The dashed curve is the corresponding function for an independent event process with the same intensity.

11. COMPARISON TO POISSON STATISTICS

If the function $f(t)$ would be a constant, e.g., $f(t) = f(\infty) = n_e$, then the intensity correlation functions would be $I_k(t_1, \ldots, t_k; \alpha) = (\alpha A n_e)^k$, and the detection process would be a stationary Poisson process with intensity $I = \alpha A n_e$. The probabilities would then be

$$P_n(0, T; \alpha) = \frac{(IT)^n}{n!} e^{-IT}. \tag{98}$$

For resonance fluorescence, the functions $P_n(0, T; 1)$ are given in the previous section, but the resulting expressions are rather formidable. We now compare graphically the functions $P_n(0, T; 1)$ for resonance fluorescence with the corresponding functions for a Poisson process with the same intensity.

The solid curve in Fig. 2 shows $P_0(0, T; 1)$ for fluorescent photons, with $\hat{\Omega} = 0.4$, and the dashed curve is the corresponding function for independent photons. The behavior of both functions is very similar for this value of $\hat{\Omega}$. For larger values of $\hat{\Omega}$, the curves become even closer, and any difference with an independent event process disappears. For $\hat{\Omega}$ large, the function $f(t)$ oscillates rapidly, as can be seen in Fig. 1, but these oscillations do not appear in the probability $P_0(0, T; 1)$.

Figure 3 shows $P_1(0, T; 1)$, the probability for the emission of one photon in $[0, T]$, for $\hat{\Omega} = 0.4$. We notice that the difference with Poisson statistics is much greater than in Fig. 2

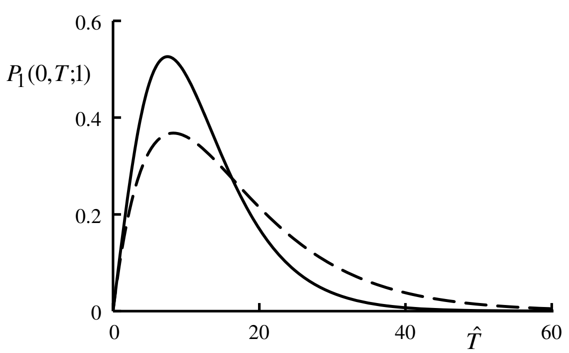

Fig. 3. Graph compares the probability for the detection of one photon from resonance fluorescence (solid curve), with $\hat{\Omega} = 0.4$, to the same probability for a Poisson process with the same intensity.

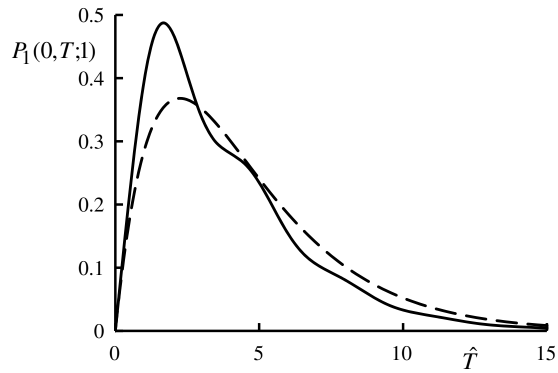

Fig. 4. Solid curve is the probability to detect one photon in $[0, T]$ for resonance fluorescence, with $\hat{\Omega} = 2$ and $\alpha = 1$. The dashed curve is the corresponding probability for a Poisson process.

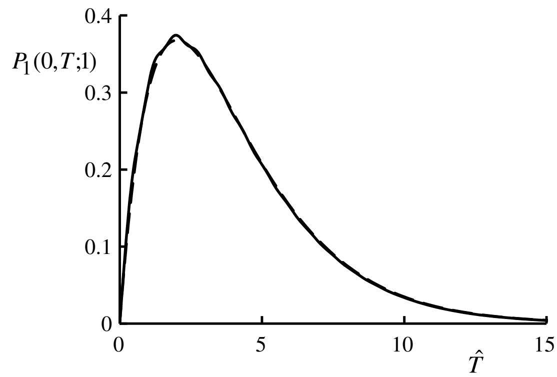

Fig. 5. Graph illustrates that for high laser power ($\hat{\Omega}$ large, and $\hat{\Omega} = 8$ for the graph), photon detection from resonance fluorescence becomes indistinguishable from a Poisson process.

(same value of $\hat{\Omega}$). Figure 4 shows $P_1(0, T; 1)$ for $\hat{\Omega} = 2$, and here we see some oscillations, due to the oscillations in the function $f(t)$ (curve b in Fig. 1). For $\hat{\Omega} = 8$, the function $f(t)$ is curve c in Fig. 1, and this function oscillates rapidly. The probability $P_1(0, T; 1)$ for this value of $\hat{\Omega}$ is shown in Fig. 5, and we see that the probability for the detection of a fluorescent photon in $[0, T]$ is almost indistinguishable from the corresponding Poissonian result. The solid curve has a tiny oscillation near its peak, but apparently the large oscillations in $f(t)$ do not appear in the probability $P_1(0, T; 1)$.

12. CONCLUSIONS

The detection of photons by a photomultiplier can be considered as a random event process. The statistical properties of such a process can be represented by a variety of functions. The most fundamental representation is in terms of intensity correlation functions. From these functions, the probability for the detection of n photons in a time interval $[t_a, t_b]$ can be obtained, as well as the probability density function for the detection of the nth photon at time t, after an initial time t_a. Also, the conditional probabilities and the conditional probability densities can be found. The condition here is the detection of a photon in the time interval $[t_a - dt_a, t_a]$.

We consider an attenuator being placed in front of the detector, which reduces the intensity by a factor α, and we compare the statistical functions for photon detection from the attenuated field to the statistical functions for photon detection from the original field. It appears that these functions are related in simple ways. The statistical functions for $\alpha < 1$ can be expressed in terms of the same functions for $\alpha = 1$. Therefore, the photon statistics for the attenuated field are known as soon as the photon statistics for the original field are known [and vice versa: the relations can be inverted, as is most easily seen from Eq. (27)]. We have also derived some interesting recurrence relations involving the α dependence of the various functions.

When photons are detected from resonance fluorescence, emitted by a two-state atom in a laser beam, then an overall factor of α^k appears in the intensity correlations for the detected photons. The parameter α appears here as the probability that an emitted photon is detected, rather than resulting from attenuation of the field. It is shown that the probability for the detection of n photons can be obtained for $\alpha = 1$, and with Eq. (30) we can then, in principle, find the probabilities for $\alpha < 1$. If we would attempt to compute the probabilities for $\alpha < 1$ directly, we would need to factor a cubic equation. For $\alpha = 1$, we only get a quadratic equation to be factored, and the result for the probabilities is given in Section 10.

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