Redistribution of energy flow in a material due to damping

Xin Li,[1,2] Donna M. Pierce,[1,3] and Henk F. Arnoldus[1,*]

[1]Department of Physics and Astronomy, Mississippi State University, P.O. Box 5167, Mississippi State, Mississippi 39762-5167, USA

[2]e-mail: xl121@msstate.edu

[3]e-mail: dmp149@msstate.edu

*Corresponding author: hfa1@msstate.edu

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The field lines of energy flow of the radiation emitted by a linear dipole in free space are straight lines, running radially outward from the source. When the dipole is embedded in a medium, the field lines are curves when the imaginary part of the relative permittivity is finite. It is shown that due to the damping in the material all radiation is emitted in directions perpendicular to the dipole axis, whereas for a dipole in free space the radiation is emitted in all directions except along the dipole axis. It is also shown that some field lines in the near field form semiloops. Energy flowing along these semiloops is absorbed by the material and does not contribute to the radiative power in the far field.

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The electromagnetic properties of a linear isotropic homogeneous material can be accounted for by the relative permittivity ε_r and the relative permeability μ_r, both of which are complex in general. The index of refraction n of the medium is a solution of $n^2 = \varepsilon_r \mu_r$, and we take the solution with $\mathrm{Im}n \geq 0$. We shall consider monochromatic radiation, with angular frequency ω, which propagates through the material. The values of ε_r, μ_r, and n depend on ω. The electric field can be written as $\mathbf{E}(\mathbf{r},t) = \mathrm{Re}[\mathbf{E}(\mathbf{r})\exp(-i\omega t)]$, with $\mathbf{E}(\mathbf{r})$ the complex amplitude, and the magnetic field $\mathbf{B}(\mathbf{r},t)$ can be represented similarly. The time-averaged Poynting vector is defined as

$$\mathbf{S}(\mathbf{r}) = \frac{1}{2\mu_o}\mathrm{Re}\left[\frac{1}{\mu_r}\mathbf{E}(\mathbf{r})^* \times \mathbf{B}(\mathbf{r})\right]. \tag{1}$$

At the field point $\mathbf{r}$, the electromagnetic energy flows in the direction of $\mathbf{S}(\mathbf{r})$, and therefore the field lines of energy flow are the field lines of the vector field $\mathbf{S}(\mathbf{r})$.

The material supports plane-wave solutions of Maxwell's equations. For a wave traveling along the z axis, we have

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_o e^{ink_o z}, \qquad \mathbf{B}(\mathbf{r}) = \frac{n}{c}\,(\mathbf{e}_z \times \mathbf{E}_o)e^{ink_o z}, \tag{2}$$

with $\mathbf{E}_o$ a vector in the xy plane and $k_o = \omega/c$. When $\mathrm{Im}n > 0$, the waves $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ decay in amplitude in the positive z direction. The Poynting vector is found to be

$$\mathbf{S}(\mathbf{r}) = \frac{1}{2\mu_o c}(\mathbf{E}_o^* \cdot \mathbf{E}_o)e^{-2k_o z \mathrm{Im}n}\mathbf{e}_z \mathrm{Re}(n/\mu_r). \tag{3}$$

It can be shown that [1] $\mathrm{Re}(n/\mu_r) \geq 0$ and therefore $\mathbf{S}(\mathbf{r})$ is in the positive z direction for all $\mathbf{r}$. The power flows in the positive z direction, and the field lines of $\mathbf{S}(\mathbf{r})$ are straight lines, parallel to the z axis. Because of the factor $\exp(-2k_o z\mathrm{Im}n)$, the magnitude of $\mathbf{S}(\mathbf{r})$ decays exponentially along a field line, so energy is dissipated upon propagation. The field lines are unaffected by the damping and are straight lines everywhere. In this Letter we shall show that in general this is not the case, and that the damping can result in a redistribution, or redirection, of the energy flow.

We shall consider the radiation emitted by a linear electric dipole, located at the origin of coordinates, oscillating along the z axis, and embedded in a material with ε_r and μ_r. The electric dipole moment is $\mathbf{d}(t) = d_o \mathbf{e}_z \cos(\omega t)$. The complex amplitudes of the emitted electric and magnetic fields can be found from a slight generalization of [2] (p. 411), and the result is

$$\mathbf{E}(\mathbf{r}) = -\mu_r \frac{d_o k_o^2}{4\pi\varepsilon_o r}\left[\mathbf{e}_\theta \sin\theta + (3\hat{\mathbf{r}}\cos\theta - \mathbf{e}_z)\frac{i}{nq}\left(1 + \frac{i}{nq}\right)\right]e^{inq}, \tag{4}$$

$$\mathbf{B}(\mathbf{r}) = -\frac{n\mu_r}{c}\frac{d_o k_o^2}{4\pi\varepsilon_o r}\mathbf{e}_\phi \sin\theta\left(1 + \frac{i}{nq}\right)e^{inq}, \tag{5}$$

where (r,θ,ϕ) are the spherical coordinates of the field point. Here we have set $q = k_o r$ for the dimensionless distance between the field point $\mathbf{r}$ and the location of the dipole. On this scale, a distance of 2π corresponds to one free-space wavelength. Because of the factors $\exp(inq)$, the electric and magnetic fields are outgoing spherical waves, and the amplitudes decay in the outgoing direction. The Poynting vector can be computed from Eq. (1). We split off an overall positive factor as

$$\mathbf{S}(\mathbf{r}) = \frac{3P_o}{8\pi r^2}|\mu_r|^2 e^{-2q\mathrm{Im}n}\boldsymbol{\sigma}(\mathbf{q}), \tag{6}$$

where P_o equals the power that would be emitted by the same dipole in free space. Vector $\boldsymbol{\sigma}$ only depends on the dimensionless position vector $\mathbf{q} = k_o \mathbf{r}$. Since the field lines of energy flow are determined only by the direction

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of $\mathbf{S}(\mathbf{r})$, and not its magnitude, the field lines of $\boldsymbol{\sigma}(\mathbf{q})$ are the same as the field lines of $\mathbf{S}(\mathbf{r})$. The vector $\boldsymbol{\sigma}(\mathbf{q})$ is found to be

$$\boldsymbol{\sigma}(\mathbf{q}) = \hat{\mathbf{r}} \sin^2\theta \mathrm{Re}\left[\frac{n}{\mu_r}\left(1+\frac{i}{nq}\right)\right] + \frac{1}{|n|^2 q}\left|1+\frac{i}{nq}\right|^2 [\hat{\mathbf{r}}(1-3\cos^2\theta) + 2\mathbf{e}_z\cos\theta]\mathrm{Im}\varepsilon_r. \tag{7}$$

For an electric dipole radiating in free space we have $\varepsilon_r = \mu_r = n = 1$, and Eq. (7) simplifies to $\boldsymbol{\sigma}(\mathbf{q}) = \hat{\mathbf{r}}\sin^2\theta$. The Poynting vector runs radially outward at all field points. Therefore, the field lines of $\boldsymbol{\sigma}(\mathbf{q})$ are straight lines, running radially outward from the dipole. Figure 1 shows the field line pattern for the emission in free space.

When the dipole is embedded in a medium, the factor Re[…] appears in the first term on the right-hand side of Eq. (7). It can be shown that this factor is positive, and therefore this term alone would give the same field line pattern as in Fig. 1. Furthermore, in the far field (q large) we have $\boldsymbol{\sigma}(\mathbf{q}) \approx \hat{\mathbf{r}}\sin^2\theta\mathrm{Re}(n/\mu_r)$, and the corresponding field lines are approximately in the radially outward direction. When the imaginary part of ε_r is nonzero, the second term on the right-hand side of Eq. (7) contributes to the Poynting vector. This term dominates in the near field (q small), and it has a part which contains $\mathbf{e}_z$. This part is responsible for a deviation of the field lines from the radially outgoing pattern of Fig. 1. Since this part only contributes when $\mathrm{Im}\varepsilon_r \neq 0$, any deviation from the radial pattern is a result of damping.

The vector field $\boldsymbol{\sigma}(\mathbf{q})$, given by Eq. (7), is rotationally symmetric around the z axis and reflection symmetric in the xy plane. Therefore we only need to consider the field lines in the first quadrant of the yz plane. In this quadrant, $\cos\theta$ is positive, and therefore the part of $\boldsymbol{\sigma}(\mathbf{q})$ containing vector $\mathbf{e}_z$ is in the positive z direction. As a result, the field lines will bend away from the radial direction, and upward. The factor in square brackets in Eq. (7) can also be written as

$$\hat{\mathbf{r}}(1-3\cos^2\theta) + 2\mathbf{e}_z\cos\theta = \sin\theta[\mathbf{e}_y(1-3\cos^2\theta) + 3\mathbf{e}_z\sin\theta\cos\theta]. \tag{8}$$

In this form we notice that the y component vanishes for $\cos\theta = 1/\sqrt{3}$, so for $\theta = 54.7°$. Therefore, under an angle of 54.7° with the z axis, the part of the Poynting vector that contains $\mathrm{Im}\varepsilon_r$ is in the positive z direction. Consequently, the field lines in the near field cross the line $\theta = 54.7°$ in a vertical, upward direction. Furthermore, for field points with a smaller angle θ, the y component of $\boldsymbol{\sigma}(\mathbf{q})$ is negative, and this means that the field line through such a point is headed toward the z axis. At the z axis we have $\theta = 0$, and it follows from Eq. (8) that the z component vanishes, relative to the y component, and therefore each field line in the near field approaches the z axis under 90°. The resulting field line pattern is illustrated in Fig. 2. The dimensionless coordinates on the axes are defined as $\bar{y} = k_o y$ and $\bar{z} = k_o z$. At larger distances, the first term on the right-hand side of Eq. (7) will contribute, and since this term is proportional to $\hat{\mathbf{r}}$, the field lines will have a tendency to bend toward the radially outward direction. This can also be seen in Fig. 2. Figure 3 shows the field lines from a wider view, and we see clearly that the field lines approach straight lines, approximately in the radial direction, except in the neighborhood of the z axis.

A field line can be parametrized as $\mathbf{q}(u)$, with u a dummy variable, and since at any point on a field line the vector $\boldsymbol{\sigma}(\mathbf{q})$ is on its tangent line, field lines are a solution of the differential equation [3] $d\mathbf{q}/du = \boldsymbol{\sigma}(\mathbf{q})$. The field lines in Figs. 2 and 3 are made by numerically solving this equation. In Cartesian coordinates, the x component of

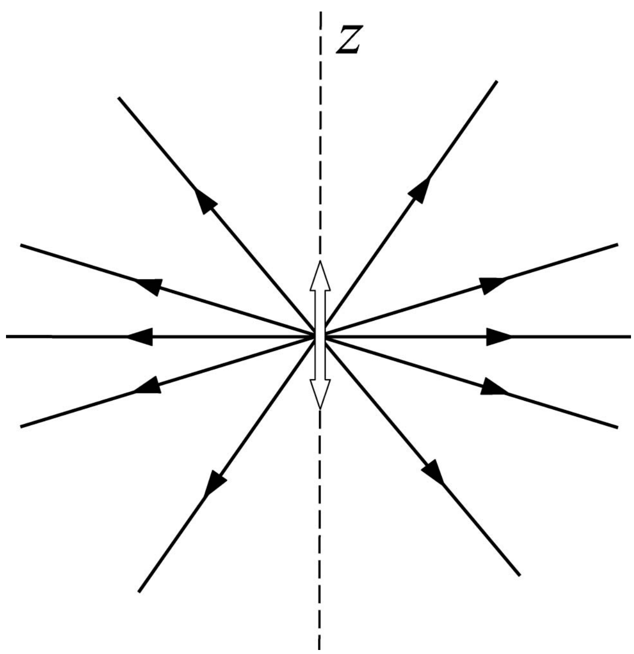

Fig. 1. Field lines of the Poynting vector of the radiation emitted by a dipole in free space, oscillating along the z axis, are straight lines in a radially outward direction.

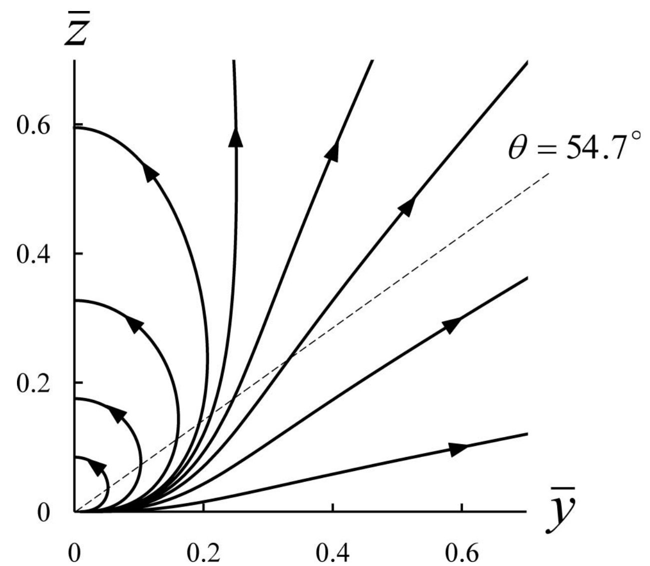

Fig. 2. Field lines of the Poynting vector for a dipole oscillating along the z axis and embedded in a material with $\varepsilon_r = 1.7 + 0.06i$ and $\mu_r = 1$. These are the values for water at 3 μm. The index of refraction is $n = 1.3 + 0.023i$.

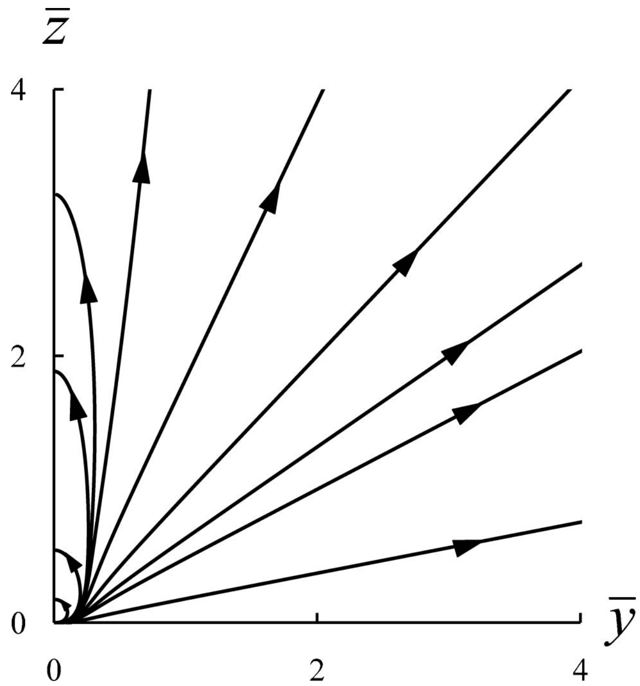

Fig. 3. Larger view of the field lines in Fig. 2. Far away from the dipole the field lines run approximately in a radial direction, except near the z axis. The curving close to the z axis is a near-field effect that persists in the far field.

this equation becomes $d\bar{x}/du = \sigma(\mathbf{q})_x$, and the same applies to the y and z components. For field lines in the yz plane, we can see $\bar{z}$ as a function of $\bar{y}$, and upon eliminating the variable u we find

$$\frac{d\bar{z}}{d\bar{y}} = \frac{\sigma(\mathbf{q})_z}{\sigma(\mathbf{q})_y} \tag{9}$$

as the equation for the field lines. The y and z components of the Poynting vector can be found using Eq. (7). We now consider field lines in the near field, for which the second term on the right-hand side of Eq. (7) dominates when $\mathrm{Im}\varepsilon_r \neq 0$. In this region, Eq. (9) becomes

$$\frac{d\bar{z}}{d\bar{y}} = \frac{3\bar{y}\bar{z}}{\bar{y}^2 - 2\bar{z}^2}. \tag{10}$$

For $\bar{y} = \bar{z}\sqrt{2}$ the right-hand side goes to infinity, and therefore the tangent line of the field line is vertical. This corresponds to the field line crossing the line $\theta = 54.7°$ in Fig. 2. Close to the $\bar{z}$ axis, the right-hand side vanishes, and therefore field lines approach the $\bar{z}$ axis horizontally. The solutions of Eq. (10) are the semiloops in Figs. 2 and 3 in the vicinity of the dipole. The energy that flows along such a semiloop comes out of the dipole and is then entirely dissipated in the material. A field line starts at the location of the dipole, below the line $\bar{y} = \bar{z}\sqrt{2}$, and it follows from Eq. (10) that near the dipole the right-hand side vanishes for $\bar{z}/\bar{y} \to 0$. Therefore, all field lines start off horizontally. In physical terms this means that all energy is emitted along the xy plane. After propagation over a finite distance, the direction of energy flow curves away from the xy plane. Some field lines form semiloops and some run to the far field, where eventually they run approximately in the radial direction, as in Fig. 1.

From Eqs. (7) and (8) we see that on the z axis the Poynting vector vanishes ($\sin\theta = 0$), and therefore the z axis is a singular line. The first term in Eq. (7) is the term that survives in the far field, and the second term is responsible for the curving of the field lines and the semiloops in the near field. The first term is proportional to $\sin^2\theta$ and the second term is proportional to $\sin\theta$. Therefore, near the z axis the far-field term goes to zero faster than the near-field term, and this holds at any distance. Since the near-field term is perpendicular to the z axis, a field line near the z axis approaches the z axis under 90°, and such a field line ends at the z axis. This is in sharp contrast to the situation without damping, as in Fig. 1, where the field lines near the z axis run parallel to the z axis. Figure 3 illustrates that indeed at large distances (as compared to the dimension of the semiloops) the field lines near the z axis bend toward the z axis and end there.

When a linear dipole is embedded in a medium, the field lines of energy flow are curves, rather than straight lines, when the imaginary part of the relative permittivity is finite. Because of the damping, the energy flow is redistributed in the material. The effect of the dissipation is not only a weakening of the power transported along a field line, as for a plane wave, but the absorption during propagation results in a dramatic change in the direction of power flow in the near field.

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