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Current densities in an illuminated perfectly-conducting sheet

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When electromagnetic radiation illuminates a perfectly-conducting metal, it induces a current density on its surface. We consider a thin perfectly-conducting sheet, possibly finite in extent and possibly with apertures in it, so that incident radiation will induce a current density at both sides of the material. The sheet current density is the sum of the current densities at both sides, and this effective current density generates the scattered field. We show that when this sheet current density is known, its splitting in surface current densities at both sides of the sheet is unique, and determined by the incident field in a simple way. A set of two coupled equations for this sheet current density is derived, which holds for any spatial structure of the incident field. This approach to scattering by a sheet is illustrated by considering a plane wave incident on a mirror and on the Sommerfeld half-plane.

Keywords: surface current density; scattering; reflection; diffraction; half-plane

1. Introduction

A layer made of perfectly-conducting metal with a flat surface and infinite in extent is a mirror. An incident plane wave reflects in the specular direction, and a light wave with an arbitrary spatial structure, which is emitted by a source in front of the mirror, reflects as if the reflected field was emitted by a mirror source behind the surface. By employing an angular spectrum representation of the source field, the current density of this image source can be constructed explicitly [1]. From a different point of view, a field incident upon a material induces a current density in the medium. For a perfect conductor, the incident light does not penetrate the metal, and therefore all induced current is a surface current which appears at the illuminated side of the metal. This surface current density emits an electromagnetic field, which is the reflected field. Let the magnetic component of the incident field have a harmonic time dependence with angular frequency ω, e.g.

$$\boldsymbol{B}(\boldsymbol{r},t)_{\mathrm{inc}} = \mathrm{Re}[\boldsymbol{B}(\boldsymbol{r})_{\mathrm{inc}}\exp(-\mathrm{i}\omega t)], \tag{1}$$

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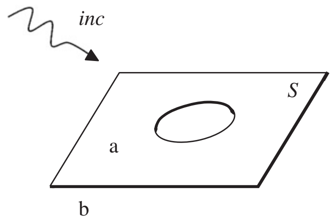

Figure 1. Radiation is incident upon a sheet S of perfectly-conducting metal. The sheet may have apertures and may be finite in extent, and therefore the illuminating light can diffract around the various edges. The radiation is incident upon the a-side of the material and the b-side is the shadow side of the sheet.

with $\boldsymbol{B}(\boldsymbol{r})_{\text{inc}}$ the complex amplitude. Other time-dependent quantities will then have the same time dependence. It can then be shown [2] that the induced surface current density $\boldsymbol{i}(\boldsymbol{r})$ at point $\boldsymbol{r}$ on the surface of the mirror is given by

$$\boldsymbol{i}(\boldsymbol{r}) = -\frac{2}{\mu_{\text{o}}}\boldsymbol{e}_z \times \boldsymbol{B}(\boldsymbol{r})_{\text{inc}}, \tag{2}$$

with $\boldsymbol{e}_z$ the unit normal on the surface, directed from the vacuum into the material. A rather remarkable feature is that the current density $\boldsymbol{i}(\boldsymbol{r})$ in a mirror at point $\boldsymbol{r}$ is determined entirely by the incident magnetic field $\boldsymbol{B}(\boldsymbol{r})_{\text{inc}}$ at that same point $\boldsymbol{r}$. For a perfect conductor of arbitrary shape, the current density at point $\boldsymbol{r}$ depends on the current density at all other points $\boldsymbol{r}'$ of the surface. The equation known as the Magnetic Field Integral Equation is a linear integral equation for the current density $\boldsymbol{i}(\boldsymbol{r})$ on the surface, and it has the incident magnetic field at the surface as an inhomogeneous term [3]. This equation, which is due to Maue [4], is widely used in numerical calculations for scattering of electromagnetic radiation off objects of arbitrary shape [5,6], and when applied to a mirror it yields immediately the result (2).

The mechanism of image formation and reflection by a mirror is independent of the thickness of the metal layer, provided we adopt the limit of perfect conductivity. In particular, it holds for an infinite sheet of material which is very thin. We shall consider a flat thin sheet S of perfectly-conducting metal, as in Figure 1. The sheet may be finite in extent and it may have one or more apertures in it. In addition to its specular reflection, the incident radiation will then pass through the aperture and diffract around the edges, giving rise to an electromagnetic field at the dark side of the screen (side b in Figure 1). The field at the illuminated side (a-side) will induce a current density $\boldsymbol{i}_{\text{a}}(\boldsymbol{r})$ in the surface, similar to the case of the mirror, and in addition, the field at the b-side will induce a surface current density $\boldsymbol{i}_{\text{b}}(\boldsymbol{r})$ at that side of the material. The situation, however, is now much more complicated because the current densities at both sides will influence each other, and these current densities will not be given by a simple relation as Equation (2). Since the sheet is very thin, the effective surface current density $\boldsymbol{i}(\boldsymbol{r})$ at point $\boldsymbol{r}$ of S is

$$\boldsymbol{i}(\boldsymbol{r}) = \boldsymbol{i}_{\text{a}}(\boldsymbol{r}) + \boldsymbol{i}_{\text{b}}(\boldsymbol{r}) \tag{3}$$

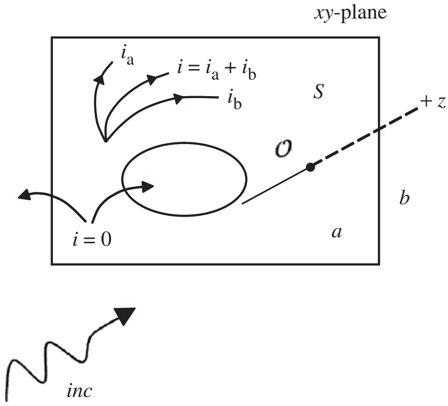

Figure 2. The incident radiation induces a surface current density $\boldsymbol{i}_a(\boldsymbol{r})$ on the a-side of the sheet S, and due to diffraction around the edges, there will be a current density $\boldsymbol{i}_b(\boldsymbol{r})$ at the b-side as well. The sheet current density $\boldsymbol{i}(\boldsymbol{r})$ is the sum of both current densities, as shown in the diagram. It is also indicated that $\boldsymbol{i}(\boldsymbol{r}) = 0$ inside an aperture and outside of the sheet. This simple observation leads to the condition expressed by Equation (43). The sheet S is part of the xy-plane, and the z-axis is directed from the a-side to the b-side. The origin of coordinates $\mathcal{O}$ can be chosen arbitrarily in the plane of the sheet.

as illustrated in Figure 2. This total current density is sometimes referred to as the sheet current density [7]. We shall derive an integral equation for $\boldsymbol{i}(\boldsymbol{r})$, which has the incident electric field at the surface as an inhomogeneous term. The equation is reminiscent of the Magnetic Field Integral Equation in that it is an equation for the current density in terms of the given incident field. However, the Magnetic Field Integral Equation is formulated in terms of the physical current densities $\boldsymbol{i}_a(\boldsymbol{r})$ and $\boldsymbol{i}_b(\boldsymbol{r})$ on the separate surfaces of the material, and its application to the situation where the separation between the two current densities becomes infinitesimally small is nontrivial [8]. The integral equation for $\boldsymbol{i}(\boldsymbol{r})$ involves the values of $\boldsymbol{i}(\boldsymbol{r}')$ at all other points $\boldsymbol{r}'$ on the surface, and the solution of this equation will also not be as simple as the solution given by Equation (2) for a mirror. However, we shall show that for a sheet S of arbitrary geometry and an incident field of arbitrary spatial dependence a generalization of Equation (2) can be derived.

2. Boundary conditions

The electromagnetic field is the sum of the incident (inc) field and the field generated by $\boldsymbol{i}(\boldsymbol{r})$, which we shall call the scattered (sc) field. For the total magnetic field we then write

$$\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{B}(\boldsymbol{r})_{\text{inc}} + \boldsymbol{B}(\boldsymbol{r})_{\text{sc}} \tag{4}$$

and similarly for the total electric field. The xy-plane is taken as the plane containing the sheet S, and the positive z-axis is directed from the a-side to the b-side, as depicted in Figure 2. The general boundary condition for the magnetic field at an interface relates the values of $\boldsymbol{B}(\boldsymbol{r})$ at two sides of the interface to the surface current density on the interface.

Inside the material of the thin sheet there is no magnetic field, so when considering a point $\boldsymbol{r}$ in the sheet the boundary condition at the a-side can be written as

$$\boldsymbol{B}(\boldsymbol{r})_{\mathrm{a}} = -\mu_{\mathrm{o}}\boldsymbol{i}_{\mathrm{a}}(\boldsymbol{r}) \times \boldsymbol{e}_z, \tag{5}$$

where it is understood that $\boldsymbol{B}(\boldsymbol{r})_{\mathrm{a}}$ is the value of the magnetic field near point $\boldsymbol{r}$ of the sheet, and just outside the medium. The boundary condition at the b-side is

$$\boldsymbol{B}(\boldsymbol{r})_{\mathrm{b}} = \mu_{\mathrm{o}}\boldsymbol{i}_{\mathrm{b}}(\boldsymbol{r}) \times \boldsymbol{e}_z. \tag{6}$$

The usual boundary conditions [9] at an interface are formulated in terms of the parallel (∥) and perpendicular (⊥) components with respect to the interface. In Equation (5), $\boldsymbol{i}_{\mathrm{a}}(\boldsymbol{r}) \times \boldsymbol{e}_z$ is a vector in the xy-plane, so it has no perpendicular component. Therefore, Equation (5) is equivalent to the set of two equations $\boldsymbol{B}(\boldsymbol{r})_{\mathrm{a},\perp} = 0$ and $\boldsymbol{B}(\boldsymbol{r})_{\mathrm{a},\parallel} = -\mu_{\mathrm{o}}\boldsymbol{i}_{\mathrm{a}}(\boldsymbol{r}) \times \boldsymbol{e}_z$, and similarly, Equation (6) can be written as a set of two equations. The boundary conditions for the electric field at the a-side and b-side are, respectively,

$$\boldsymbol{E}(\boldsymbol{r})_{\mathrm{a}} = -\frac{\sigma_{\mathrm{a}}(\boldsymbol{r})}{\varepsilon_{\mathrm{o}}}\boldsymbol{e}_z, \tag{7}$$

$$\boldsymbol{E}(\boldsymbol{r})_{\mathrm{b}} = \frac{\sigma_{\mathrm{b}}(\boldsymbol{r})}{\varepsilon_{\mathrm{o}}}\boldsymbol{e}_z, \tag{8}$$

which involve the surface charge densities $\sigma_{\mathrm{a}}(\boldsymbol{r})$ and $\sigma_{\mathrm{b}}(\boldsymbol{r})$. These equations can also be split into their parallel and perpendicular components, so that Equations (7) and (8) become a set of four equations.

In the boundary condition $\boldsymbol{B}(\boldsymbol{r})_{\mathrm{a},\perp} = 0$, which follows from Equation (5), the magnetic field is the total magnetic field just outside the sheet S, and at the a-side. When we split the field as in Equation (4), this boundary condition can also be written as

$$\boldsymbol{B}(\boldsymbol{r})_{\mathrm{a,sc},\perp} = -\boldsymbol{B}(\boldsymbol{r})_{\mathrm{inc},\perp}, \tag{9}$$

relating the perpendicular component of the unknown scattered field at the a-side to the perpendicular component of the given incident field at the same point. For the incident field on the right-hand side we have dropped the subscript 'a', since the incident field is continuous across the sheet. In the same way we find from Equation (7)

$$\boldsymbol{E}(\boldsymbol{r})_{\mathrm{a,sc},\parallel} = -\boldsymbol{E}(\boldsymbol{r})_{\mathrm{inc},\parallel}. \tag{10}$$

Then, Equations (9) and (10) also hold if we replace 'a' by 'b', as follows from Equations (6) and (8), and therefore we obtain

$$\boldsymbol{B}(\boldsymbol{r})_{\mathrm{a,sc},\perp} = \boldsymbol{B}(\boldsymbol{r})_{\mathrm{b,sc},\perp}, \tag{11}$$

$$\boldsymbol{E}(\boldsymbol{r})_{\mathrm{a,sc},\parallel} = \boldsymbol{E}(\boldsymbol{r})_{\mathrm{b,sc},\parallel}. \tag{12}$$

Interestingly, the perpendicular (parallel) components of the scattered magnetic (electric) field are the same at both sides of the sheet, even though the current densities at both sides are different.

Since the current density vectors $\boldsymbol{i}_{\mathrm{a}}(\boldsymbol{r})$ and $\boldsymbol{i}_{\mathrm{b}}(\boldsymbol{r})$ lie in the xy-plane, we have the identity

$$\boldsymbol{e}_z \times [\boldsymbol{i}_{\mathrm{a,\,b}}(\boldsymbol{r}) \times \boldsymbol{e}_z] = \boldsymbol{i}_{\mathrm{a,\,b}}(\boldsymbol{r}), \tag{13}$$

which allows us to invert Equations (5) and (6) as

$$\boldsymbol{i}_{\mathrm{a}}(\boldsymbol{r}) = -\frac{1}{\mu_{\mathrm{o}}}\boldsymbol{e}_z \times \boldsymbol{B}(\boldsymbol{r})_{\mathrm{a}}, \tag{14}$$

$$\boldsymbol{i}_{\mathrm{b}}(\boldsymbol{r}) = \frac{1}{\mu_{\mathrm{o}}}\boldsymbol{e}_z \times \boldsymbol{B}(\boldsymbol{r})_{\mathrm{b}}. \tag{15}$$

These current densities are determined by the total magnetic field at the a-side and b-side, respectively. The sum of $\boldsymbol{i}_{\mathrm{a}}(\boldsymbol{r})$ and $\boldsymbol{i}_{\mathrm{b}}(\boldsymbol{r})$ is the sheet current density $\boldsymbol{i}(\boldsymbol{r})$, which becomes

$$\boldsymbol{i}(\boldsymbol{r}) = \frac{1}{\mu_{\mathrm{o}}}\boldsymbol{e}_z \times [\boldsymbol{B}(\boldsymbol{r})_{\mathrm{b,\,sc}} - \boldsymbol{B}(\boldsymbol{r})_{\mathrm{a,\,sc}}], \tag{16}$$

where we have used that the incident field is continuous across the sheet. We see that the total current density $\boldsymbol{i}(\boldsymbol{r})$ can be expressed in terms of the scattered field only.

3. The scattered field

The current density $\boldsymbol{i}(\boldsymbol{r})$ generates the scattered field, and the magnetic component is given by

$$\boldsymbol{B}(\boldsymbol{r})_{\mathrm{sc}} = \frac{\mu_{\mathrm{o}}}{4\pi}\nabla \times \int \mathrm{d}S'\boldsymbol{i}(\boldsymbol{r}')g(\boldsymbol{r}-\boldsymbol{r}'), \tag{17}$$

in terms of the Green's function for the scalar Helmholtz equation

$$g(\boldsymbol{r}-\boldsymbol{r}') = \frac{\exp(\mathrm{i}k_{\mathrm{o}}|\boldsymbol{r}-\boldsymbol{r}'|)}{|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{18}$$

where $k_{\mathrm{o}} = \omega/c$. The electric component of the scattered field follows from the magnetic component as

$$\boldsymbol{E}(\boldsymbol{r})_{\mathrm{sc}} = \frac{\mathrm{i}c^2}{\omega}\nabla \times \boldsymbol{B}(\boldsymbol{r})_{\mathrm{sc}}, \tag{19}$$

according to one of Maxwell's equations.

Particularly useful for the present problem is Weyl's representation of the Green's function [10]

$$g(\boldsymbol{r}-\boldsymbol{r}') = \frac{\mathrm{i}}{2\pi}\int \mathrm{d}^2\boldsymbol{k}_{\parallel}\frac{1}{\beta}\exp\left[\mathrm{i}\boldsymbol{k}_{\parallel}\cdot(\boldsymbol{r}-\boldsymbol{r}') + \mathrm{i}\beta|z-z'|\right]. \tag{20}$$

The integral runs over the entire $\boldsymbol{k}_\parallel$-plane, and the parameter β is defined as

$$\beta = \begin{cases} \left(k_o^2 - k_\parallel^2\right)^{1/2}, & k_\parallel < k_o, \\ \mathrm{i}\left(k_\parallel^2 - k_o^2\right)^{1/2}, & k_\parallel > k_o. \end{cases} \tag{21}$$

Equation (20) is an angular spectrum representation of the Green's function. For β real, a partial wave is a traveling wave, whereas for β imaginary the partial wave is evanescent in both the positive and negative z-directions. Since $\boldsymbol{i}(\boldsymbol{r}')$ in Equation (17) is in the xy-plane, we only need $g(\boldsymbol{r}-\boldsymbol{r}')$ for $z'=0$. We then introduce the notation

$$\boldsymbol{K}_\pm = \boldsymbol{k}_\parallel \pm \beta \boldsymbol{e}_z, \tag{22}$$

with $\pm = \mathrm{sgn}(z)$. So in the following equations we use the upper (lower) sign for the region $z>0$ ($z<0$). When we substitute Weyl's representation (20) into Equation (17) we obtain for the scattered magnetic field

$$\boldsymbol{B}(\boldsymbol{r})_{\mathrm{sc}} = -\frac{\mu_o}{8\pi^2}\int \mathrm{d}^2\boldsymbol{k}_\parallel \frac{1}{\beta}\exp(\mathrm{i}\boldsymbol{K}_\pm\cdot\boldsymbol{r})\boldsymbol{K}_\pm \times \boldsymbol{I}(\boldsymbol{k}_\parallel), \tag{23}$$

where the transformed current density $\boldsymbol{I}(\boldsymbol{k}_\parallel)$ is defined as

$$\boldsymbol{I}(\boldsymbol{k}_\parallel) = \int \mathrm{d}S\boldsymbol{i}(\boldsymbol{r})\exp(-\mathrm{i}\boldsymbol{k}_\parallel\cdot\boldsymbol{r}). \tag{24}$$

The scattered electric field then follows from Equations (19) and (23), and we find

$$\boldsymbol{E}(\boldsymbol{r})_{\mathrm{sc}} = \frac{1}{8\pi^2\varepsilon_o\omega}\int \mathrm{d}^2\boldsymbol{k}_\parallel \frac{1}{\beta}\exp(\mathrm{i}\boldsymbol{K}_\pm\cdot\boldsymbol{r})\boldsymbol{K}_\pm \times [\boldsymbol{K}_\pm \times \boldsymbol{I}(\boldsymbol{k}_\parallel)]. \tag{25}$$

4. The scattered field near the sheet

It should be noted that the scattered magnetic and electric fields, given by Equations (23) and (25), respectively, are a solution of Maxwell's equations for any current distribution $\boldsymbol{i}(\boldsymbol{r})$ in the sheet. For a particular problem, like a sheet with an aperture or an edge, we need to consider the boundary conditions. The sheet is in the xy-plane, so we set $z=0$, and this gives $\exp(\mathrm{i}\boldsymbol{K}_\pm\cdot\boldsymbol{r}) = \exp(\mathrm{i}\boldsymbol{k}_\parallel\cdot\boldsymbol{r})$ in Equations (23) and (25). With some vector identities we then split the magnetic and electric fields into their perpendicular and parallel components, which yields

$$\boldsymbol{B}(\boldsymbol{r})_{\mathrm{sc},\perp} = -\frac{\mu_o}{8\pi^2}\int \mathrm{d}^2\boldsymbol{k}_\parallel \frac{1}{\beta}\exp(\mathrm{i}\boldsymbol{k}_\parallel\cdot\boldsymbol{r})\boldsymbol{k}_\parallel \times \boldsymbol{I}(\boldsymbol{k}_\parallel), \tag{26}$$

$$\boldsymbol{B}(\boldsymbol{r})_{\mathrm{sc},\parallel} = \mp\frac{\mu_{\mathrm{o}}}{8\pi^2}\boldsymbol{e}_z \times \int \mathrm{d}^2\boldsymbol{k}_\parallel \exp(\mathrm{i}\boldsymbol{k}_\parallel\cdot\boldsymbol{r})\boldsymbol{I}(\boldsymbol{k}_\parallel), \tag{27}$$

$$\boldsymbol{E}(\boldsymbol{r})_{\mathrm{sc},\perp} = \pm\frac{1}{8\pi^2\varepsilon_{\mathrm{o}}\omega}\boldsymbol{e}_z \int \mathrm{d}^2\boldsymbol{k}_\parallel \exp(\mathrm{i}\boldsymbol{k}_\parallel\cdot\boldsymbol{r})\boldsymbol{k}_\parallel\cdot\boldsymbol{I}(\boldsymbol{k}_\parallel), \tag{28}$$

$$\boldsymbol{E}(\boldsymbol{r})_{\mathrm{sc},\parallel} = \frac{1}{8\pi^2\varepsilon_{\mathrm{o}}\omega}\int \mathrm{d}^2\boldsymbol{k}_\parallel \frac{1}{\beta}\exp(\mathrm{i}\boldsymbol{k}_\parallel\cdot\boldsymbol{r})\{\boldsymbol{k}_\parallel[\boldsymbol{k}_\parallel\cdot\boldsymbol{I}(\boldsymbol{k}_\parallel)] - k_{\mathrm{o}}^2\boldsymbol{I}(\boldsymbol{k}_\parallel)\}, \tag{29}$$

for the scattered fields just off the sheet. The $\mp$ and $\pm$ signs in Equations (27) and (28) refer to the two sides of the screen, with the upper (lower) sign holding for the a- (b)- side. Therefore, $\boldsymbol{B}(\boldsymbol{r})_{\mathrm{sc},\parallel}$ and $\boldsymbol{E}(\boldsymbol{r})_{\mathrm{sc},\perp}$ are discontinuous across the screen. Equations (26) and (29) do not have such a $\pm$ sign, and therefore $\boldsymbol{B}(\boldsymbol{r})_{\mathrm{sc},\perp}$ and $\boldsymbol{E}(\boldsymbol{r})_{\mathrm{sc},\parallel}$ are the same for the a-side and the b-side. This is just what is expressed by Equations (11) and (12), so these boundary conditions are fulfilled automatically for any current density $\boldsymbol{i}(\boldsymbol{r})$ in the sheet.

The appearance of the $\mp$ sign in Equation (27) implies that

$$\boldsymbol{B}(\boldsymbol{r})_{\mathrm{b,sc},\parallel} = -\boldsymbol{B}(\boldsymbol{r})_{\mathrm{a,sc},\parallel}. \tag{30}$$

In Equation (16) we can replace $\boldsymbol{B}(\boldsymbol{r})_{\mathrm{a(b),sc}}$ by $\boldsymbol{B}(\boldsymbol{r})_{\mathrm{a(b),sc},\parallel}$ since $\boldsymbol{e}_z\times\boldsymbol{B}(\boldsymbol{r})_{\mathrm{a(b),sc},\perp}=0$. Then we substitute the right-hand side of Equation (30) for $\boldsymbol{B}(\boldsymbol{r})_{\mathrm{b,sc},\parallel}$ and then we drop the $\parallel$ again. This yields for the sheet current density

$$\boldsymbol{i}(\boldsymbol{r}) = -\frac{2}{\mu_{\mathrm{o}}}\boldsymbol{e}_z\times\boldsymbol{B}(\boldsymbol{r})_{\mathrm{a,sc}}. \tag{31}$$

It is interesting to compare this expression to the result (2) for a mirror. The surface current density on the mirror is determined by the local incident field, whereas the surface current density at the illuminated side of a sheet is determined by the local scattered field by an otherwise identical expression.

The transformed current density, $\boldsymbol{I}(\boldsymbol{k}_\parallel)$, is defined by Equation (24). This equation can be inverted as

$$\frac{1}{4\pi^2}\int \mathrm{d}^2\boldsymbol{k}_\parallel \exp(\mathrm{i}\boldsymbol{k}_\parallel\cdot\boldsymbol{r})\boldsymbol{I}(\boldsymbol{k}_\parallel) = \begin{cases}\boldsymbol{i}(\boldsymbol{r}), & \boldsymbol{r}\text{ in S},\\ 0, & \boldsymbol{r}\text{ not in S},\end{cases} \tag{32}$$

for $\boldsymbol{r}$ in the xy-plane. This follows from the representation of the two-dimensional delta function

$$\frac{1}{4\pi^2}\int \mathrm{d}^2\boldsymbol{k}_\parallel \exp[\mathrm{i}\boldsymbol{k}_\parallel\cdot(\boldsymbol{r}-\boldsymbol{r}')] = \delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{33}$$

The integral in Equation (27) is the same as in Equation (32), and therefore we find

$$\boldsymbol{B}(\boldsymbol{r})_{\mathrm{sc},\parallel} = \begin{cases}\mp\frac{1}{2}\mu_{\mathrm{o}}\boldsymbol{e}_z\times\boldsymbol{i}(\boldsymbol{r}), & \boldsymbol{r}\text{ in S},\\ 0, & \boldsymbol{r}\text{ not in S},\end{cases} \tag{34}$$

for $\boldsymbol{r}$ in the xy-plane. For $\boldsymbol{r}$ in S, the upper sign holds for the a-side, and when we take the cross product with $\boldsymbol{e}_z$ this yields again Equation (31). The sheet charge density $\sigma(\boldsymbol{r})$ follows from the sheet current density through the continuity equation

$$\sigma(\boldsymbol{r}) = -\frac{\mathrm{i}}{\omega}\nabla\cdot\boldsymbol{i}(\boldsymbol{r}) \tag{35}$$

and it is easily verified that Equation (28) can be simplified to

$$\boldsymbol{E}(\boldsymbol{r})_{\mathrm{sc},\perp} = \begin{cases} \pm\dfrac{1}{2\varepsilon_{\mathrm{o}}}\boldsymbol{e}_z\sigma(\boldsymbol{r}), & \boldsymbol{r} \text{ in S}, \\ 0, & \boldsymbol{r} \text{ not in S}, \end{cases} \tag{36}$$

for $\boldsymbol{r}$ in the xy-plane.

5. The current densities

The sheet current density can be found from the scattered magnetic field at the illuminated side, according to Equation (31). For a given incident field, however, the scattered field is an unknown. We now derive an equation between the current densities and the incident field. To this end, we first notice that in Equations (14) and (15) we can replace the magnetic fields at each side of the sheet by their parallel components, since the cross products between $\boldsymbol{e}_z$ and the perpendicular components yield zero. We then find for the difference between the current densities

$$\boldsymbol{i}_{\mathrm{a}}(\boldsymbol{r}) - \boldsymbol{i}_{\mathrm{b}}(\boldsymbol{r}) = -\frac{1}{\mu_{\mathrm{o}}}\boldsymbol{e}_z \times \left[\boldsymbol{B}(\boldsymbol{r})_{\mathrm{a},\parallel} + \boldsymbol{B}(\boldsymbol{r})_{\mathrm{b},\parallel}\right]. \tag{37}$$

At the right-hand side we find the parallel components of the total magnetic fields at sides a and b of the sheet. When we split these in their incident and scattered parts, then we see from Equation (30) that the contributions from the scattered parts cancel. The incident field at the a-side and b-side is the same, so we obtain

$$\boldsymbol{i}_{\mathrm{a}}(\boldsymbol{r}) - \boldsymbol{i}_{\mathrm{b}}(\boldsymbol{r}) = -\frac{2}{\mu_{\mathrm{o}}}\boldsymbol{e}_z \times \boldsymbol{B}(\boldsymbol{r})_{\mathrm{inc}}, \tag{38}$$

where we have left out again the subscripts ‘a’ and $\parallel$. The right-hand side of Equation (38) is identical to the right-hand side of Equation (2) for the mirror. Since the mirror is a special case of a sheet with openings and edges, Equation (38) is the generalization of Equation (2). For a mirror, the incident radiation can not go around the edges or through an aperture, so we must have $\boldsymbol{i}_{\mathrm{b}}(\boldsymbol{r}) = 0$. Current only appears at the illuminated side, and we have $\boldsymbol{i}_{\mathrm{a}}(\boldsymbol{r}) = \boldsymbol{i}(\boldsymbol{r})$. We find that for a sheet the local incident magnetic field at point $\boldsymbol{r}$ determines the difference between the current densities at the same point $\boldsymbol{r}$, rather than the sheet current density $\boldsymbol{i}(\boldsymbol{r})$.

By combining Equations (3) and (38) we can write the current densities at each side of the sheet as

$$\boldsymbol{i}_{\rm a}(\boldsymbol{r}) = \frac{1}{2}\boldsymbol{i}(\boldsymbol{r}) - \frac{1}{\mu_{\rm o}}\boldsymbol{e}_z \times \boldsymbol{B}(\boldsymbol{r})_{\rm inc}, \tag{39}$$

$$\boldsymbol{i}_{\rm b}(\boldsymbol{r}) = \frac{1}{2}\boldsymbol{i}(\boldsymbol{r}) + \frac{1}{\mu_{\rm o}}\boldsymbol{e}_z \times \boldsymbol{B}(\boldsymbol{r})_{\rm inc}. \tag{40}$$

In this form it seems that each side of the sheet acquires half of the sheet current $\boldsymbol{i}(\boldsymbol{r})$, and at each side there is a correction which is determined by the local incident field. Or, from another point of view, when the sheet current is known, its splitting in the separate $\boldsymbol{i}_{\rm a}(\boldsymbol{r})$ and $\boldsymbol{i}_{\rm b}(\boldsymbol{r})$ is determined uniquely by the local incident field and is independent of the geometry of the sheet. By writing Equation (38) as

$$\boldsymbol{i}_{\rm a}(\boldsymbol{r}) = -\frac{2}{\mu_{\rm o}}\boldsymbol{e}_z \times \boldsymbol{B}(\boldsymbol{r})_{\rm inc} + \boldsymbol{i}_{\rm b}(\boldsymbol{r}), \tag{41}$$

we arrive at yet another interpretation. The first term on the right-hand side is what the current density at the illuminated side would be if the sheet were a mirror. The second term, $\boldsymbol{i}_{\rm b}(\boldsymbol{r})$, is a correction due to apertures and edges. Since the b-side has the current density $\boldsymbol{i}_{\rm b}(\boldsymbol{r})$, which would be zero for a mirror, we conclude that the current densities at both sides of the sheet gain the same additional $\boldsymbol{i}_{\rm b}(\boldsymbol{r})$, as compared to the mirror. The sheet current density is therefore what the current density would be for a mirror, plus twice the current density at the shadow side.

6. The sheet current density

From Equations (39) and (40) we see that the current densities $\boldsymbol{i}_{\rm a}(\boldsymbol{r})$ and $\boldsymbol{i}_{\rm b}(\boldsymbol{r})$ can be found as soon as we know the sheet current density $\boldsymbol{i}(\boldsymbol{r})$. In order to obtain an equation for $\boldsymbol{i}(\boldsymbol{r})$, we first consider the boundary condition given by Equation (10). Comparison with Equation (29) gives immediately

$$\frac{1}{8\pi^2\varepsilon_{\rm o}\omega}\int {\rm d}^2\boldsymbol{k}_\parallel \frac{1}{\beta}\exp({\rm i}\boldsymbol{k}_\parallel\cdot\boldsymbol{r})\{k_{\rm o}^2\boldsymbol{I}(\boldsymbol{k}_\parallel) - \boldsymbol{k}_\parallel[\boldsymbol{k}_\parallel\cdot\boldsymbol{I}(\boldsymbol{k}_\parallel)]\} = \boldsymbol{E}(\boldsymbol{r})_{{\rm inc},\parallel}, \quad \boldsymbol{r} \text{ in S}, \tag{42}$$

a linear integral equation for the unknown transformed current density $\boldsymbol{I}(\boldsymbol{k}_\parallel)$, with the incident electric field on the right-hand side as an inhomogeneous term. The equation has $\boldsymbol{r}$ as a parameter, and the solution $\boldsymbol{I}(\boldsymbol{k}_\parallel)$ has to satisfy Equation (42) for every $\boldsymbol{r}$ in the sheet S. The left-hand side of Equation (42) has the appearance of a two-dimensional Fourier transform of a function of $\boldsymbol{k}_\parallel$, and the parameter $\boldsymbol{r}$ is the variable of the transformed function. Equation (42) is one of the boundary conditions at point $\boldsymbol{r}$ in S, but the left-hand side defines a function of $\boldsymbol{r}$ for all $\boldsymbol{r}$ in the xy-plane. When $\boldsymbol{r}$ is, for instance, a point of the aperture, obviously Equation (42) does not hold, since this equation expresses that the a-side of S is the boundary between vacuum and a perfect conductor. Therefore, we need to supplement Equation (42) with a condition relating to the transform of $\boldsymbol{I}(\boldsymbol{k}_\parallel)$ for a point $\boldsymbol{r}$ in the xy-plane, but not in S. From Equation (32) we have

$$\int \mathrm{d}^2\boldsymbol{k}_\parallel \exp(\mathrm{i}\boldsymbol{k}_\parallel \cdot \boldsymbol{r})\boldsymbol{I}(\boldsymbol{k}_\parallel) = 0, \quad \boldsymbol{r} \text{ not in S}, \tag{43}$$

which expresses that the sheet current density is zero for $\boldsymbol{r}$ in the xy-plane but not in S. The function $\boldsymbol{I}(\boldsymbol{k}_\parallel)$, defined on the entire $\boldsymbol{k}_\parallel$-plane, has to satisfy Equation (42) and (43) simultaneously.

We have not yet considered the boundary condition for the magnetic field, given by Equation (9). With Equation (26) this boundary condition becomes

$$\frac{\mu_\mathrm{o}}{8\pi^2}\int \mathrm{d}^2\boldsymbol{k}_\parallel \frac{1}{\beta}\exp(\mathrm{i}\boldsymbol{k}_\parallel \cdot \boldsymbol{r})\boldsymbol{k}_\parallel \times \boldsymbol{I}(\boldsymbol{k}_\parallel) = \boldsymbol{B}(\boldsymbol{r})_{\mathrm{inc},\perp}, \quad \boldsymbol{r} \text{ in S}, \tag{44}$$

as a possible alternative to Equation (42). However, the incident field obeys Maxwell's equation

$$\boldsymbol{B}(\boldsymbol{r})_\mathrm{inc} = -\frac{\mathrm{i}}{\omega}\nabla \times \boldsymbol{E}(\boldsymbol{r})_\mathrm{inc} \tag{45}$$

and if we apply the operation

$$\frac{\partial}{\partial x}\boldsymbol{e}_y \cdot (\ldots) - \frac{\partial}{\partial y}\boldsymbol{e}_x \cdot (\ldots) \tag{46}$$

on both sides of Equation (42), then Equation (42) goes over into Equation (44). Therefore, any solution of Equation (42) is also a solution of Equation (44). Equation (44) is dependent, and does not provide any additional information.

7. Plane wave incident upon a mirror

The solution $\boldsymbol{I}(\boldsymbol{k}_\parallel)$ of Equations (42) and (43) will depend on the incident field, due to the term $\boldsymbol{E}(\boldsymbol{r})_{\mathrm{inc},\parallel}$ on the right-hand side of Equation (42), but it will also depend on the geometry of the sheet S. Once this solution is obtained, the scattered magnetic and electric fields follow from Equations (23) and (25), respectively. The sheet current density $\boldsymbol{i}(\boldsymbol{r})$ follows from Equation (32), after which the current densities $\boldsymbol{i}_\mathrm{a}(\boldsymbol{r})$ and $\boldsymbol{i}_\mathrm{b}(\boldsymbol{r})$ can be obtained from Equations (39) and (40), respectively. In order to illustrate this approach we now consider the case where the sheet S is a mirror, and the incident field is taken as the plane wave

$$\boldsymbol{E}(\boldsymbol{r})_\mathrm{inc} = E\boldsymbol{\varepsilon}\exp(\mathrm{i}\boldsymbol{k}_\mathrm{o} \cdot \boldsymbol{r}). \tag{47}$$

The amplitude E and polarization vector $\boldsymbol{\varepsilon}$ are arbitrary and complex-valued, and the wave vector is

$$\boldsymbol{k}_\mathrm{o} = \boldsymbol{k}_{\mathrm{o},\parallel} + \beta_\mathrm{o}\boldsymbol{e}_z. \tag{48}$$

We shall assume that $\boldsymbol{k}_{o,\parallel}$ is real and given, and since $k_o = \omega/c$ is given, this yields for the z-component

$$\beta_o = \begin{cases} \left(k_o^2 - k_{o,\parallel}^2\right)^{1/2}, & k_{o,\parallel} < k_o, \\ i\left(k_{o,\parallel}^2 - k_o^2\right)^{1/2}, & k_{o,\parallel} > k_o. \end{cases} \tag{49}$$

The two possibilities correspond to a traveling and evanescent incident wave, just as in Equation (21) for the partial waves in the angular spectrum representation of the Green's function.

We only need to consider Equation (42), because the mirror covers the entire xy-plane. We substitute $E\boldsymbol{\varepsilon}_\parallel \exp(i\boldsymbol{k}_o \cdot \boldsymbol{r})$ for the right-hand side, multiply through by $\exp(-i\boldsymbol{k}'_\parallel \cdot \boldsymbol{r})$, and integrate over the xy-plane. With the representation

$$\int dS \exp[i(\boldsymbol{k}_\parallel - \boldsymbol{k}'_\parallel) \cdot \boldsymbol{r}] = 4\pi^2 \delta(\boldsymbol{k}_\parallel - \boldsymbol{k}'_\parallel) \tag{50}$$

and similarly with $\boldsymbol{k}_\parallel$ replaced by $\boldsymbol{k}_{o,\parallel}$, the integral Equation (42) goes over into

$$\frac{1}{8\pi^2 \varepsilon_o \omega} \frac{1}{\beta} \{k_o^2 \boldsymbol{I}(\boldsymbol{k}_\parallel) - \boldsymbol{k}_\parallel [\boldsymbol{k}_\parallel \cdot \boldsymbol{I}(\boldsymbol{k}_\parallel)]\} = E\boldsymbol{\varepsilon}_\parallel \delta(\boldsymbol{k}_\parallel - \boldsymbol{k}_{o,\parallel}), \tag{51}$$

after dropping the prime on $\boldsymbol{k}'_\parallel$. Equation (51) is now an algebraic equation for $\boldsymbol{I}(\boldsymbol{k}_\parallel)$. Due to the delta function on the right-hand side, we can replace the two $\boldsymbol{k}_\parallel$'s on the left-hand side by $\boldsymbol{k}_{o,\parallel}$, and β by β_o. With some vector manipulations, Equation (51) can be solved, and the result is

$$\boldsymbol{I}(\boldsymbol{k}_\parallel) = 8\pi^2 \varepsilon_o c E \delta(\boldsymbol{k}_\parallel - \boldsymbol{k}_{o,\parallel}) \boldsymbol{p}, \tag{52}$$

where we introduced the abbreviation

$$\boldsymbol{p} = \boldsymbol{e}_z \times (\boldsymbol{\varepsilon} \times \hat{\boldsymbol{k}}_o) \tag{53}$$

with $\hat{\boldsymbol{k}}_o = \boldsymbol{k}_o / k_o$. The sheet current density follows from Equation (32), and we obtain

$$\boldsymbol{i}(\boldsymbol{r}) = 2\varepsilon_o c E \boldsymbol{p} \exp(i\boldsymbol{k}_{o,\parallel} \cdot \boldsymbol{r}). \tag{54}$$

The incident magnetic field is

$$\boldsymbol{B}(\boldsymbol{r})_{\text{inc}} = \frac{E}{c} \hat{\boldsymbol{k}}_o \times \boldsymbol{\varepsilon} \exp(i\boldsymbol{k}_o \cdot \boldsymbol{r}), \tag{55}$$

which has as its polarization vector

$$\boldsymbol{\varepsilon}' = \hat{\boldsymbol{k}}_o \times \boldsymbol{\varepsilon}, \tag{56}$$

and the value of $\boldsymbol{B}(\boldsymbol{r})_{\rm inc}$ at $z=0$ follows from replacing $\exp(i\boldsymbol{k}_{\rm o}\cdot\boldsymbol{r})$ by $\exp(i\boldsymbol{k}_{{\rm o},\parallel}\cdot\boldsymbol{r})$. From Equations (53)–(55) we then find the relation

$$\boldsymbol{i}(\boldsymbol{r}) = -\frac{2}{\mu_{\rm o}}\boldsymbol{e}_z \times \boldsymbol{B}(\boldsymbol{r})_{\rm inc}, \tag{57}$$

which is the same as Equation (2). When we substitute the right-hand side into Equations (39) and (40) we see that $\boldsymbol{i}_{\rm a}(\boldsymbol{r}) = \boldsymbol{i}(\boldsymbol{r})$ and $\boldsymbol{i}_{\rm b}(\boldsymbol{r}) = 0$. Therefore, for an infinite sheet without apertures all current appears at the illuminated side, as could be expected.

In this approach, the scattered magnetic and electric fields can be evaluated easily. We substitute the result (52) for $\boldsymbol{I}(\boldsymbol{k}_\parallel)$ into Equations (23) and (25). Due to the delta function, the vector $\boldsymbol{k}_\parallel$ is replaced by $\boldsymbol{k}_{{\rm o},\parallel}$, so that β becomes $\beta_{\rm o}$ and vector $\boldsymbol{K}_\pm$ from Equation (22) becomes $\boldsymbol{k}_{{\rm o},\parallel} \pm \beta_{\rm o}\boldsymbol{e}_z$. Let us first consider the equations with the upper sign, which hold for $z > 0$. The value of $\boldsymbol{K}_+$ at $\boldsymbol{k}_{{\rm o},\parallel}$ is just the wave vector $\boldsymbol{k}_{\rm o}$ of the incident wave from Equation (48). With

$$\boldsymbol{k}_{\rm o} \times \boldsymbol{p} = \beta_{\rm o}\hat{\boldsymbol{k}}_{\rm o} \times \boldsymbol{\varepsilon}, \tag{58}$$

which can be derived from Equation (53), we find for the scattered magnetic field

$$\boldsymbol{B}(\boldsymbol{r})_{\rm sc} = -\frac{E}{c}\boldsymbol{\varepsilon}' \exp(i\boldsymbol{k}_{\rm o}\cdot\boldsymbol{r}), \quad z > 0, \tag{59}$$

and with

$$\boldsymbol{k}_{\rm o} \times (\boldsymbol{k}_{\rm o} \times \boldsymbol{p}) = -k_{\rm o}\beta_{\rm o}\boldsymbol{\varepsilon}, \tag{60}$$

the scattered electric field becomes

$$\boldsymbol{E}(\boldsymbol{r})_{\rm sc} = -E\boldsymbol{\varepsilon} \exp(i\boldsymbol{k}_{\rm o}\cdot\boldsymbol{r}), \quad z > 0. \tag{61}$$

Comparison with Equations (47) and (55) shows that the scattered fields are the opposite of the incident field and therefore the total electric and magnetic fields vanish behind the mirror. The equations with the lower sign hold for the scattered field in front of the mirror, so this is the reflected (specular) field. The value of $\boldsymbol{K}_-$ at $\boldsymbol{k}_{{\rm o},\parallel}$ is the wave vector $\boldsymbol{k}_{\rm r}$ of the reflected wave, which follows from $\boldsymbol{k}_{\rm o}$ by reversing the sign of the z-component:

$$\boldsymbol{k}_{\rm r} = \boldsymbol{k}_{{\rm o},\parallel} - \beta_{\rm o}\boldsymbol{e}_z. \tag{62}$$

The reflected fields can then be represented as

$$\boldsymbol{B}(\boldsymbol{r})_{\rm sc} = \frac{E}{c}\boldsymbol{\varepsilon}'_{\rm r} \exp(i\boldsymbol{k}_{\rm r}\cdot\boldsymbol{r}), \quad z < 0, \tag{63}$$

$$\boldsymbol{E}(\boldsymbol{r})_{\rm sc} = E\boldsymbol{\varepsilon}_{\rm r} \exp(i\boldsymbol{k}_{\rm r}\cdot\boldsymbol{r}), \quad z < 0, \tag{64}$$

and their polarization vectors are defined as

$$\boldsymbol{\varepsilon}_{\rm r}' = -\frac{1}{\beta_{\rm o}}\boldsymbol{k}_{\rm r}\times\boldsymbol{p}, \tag{65}$$

$$\boldsymbol{\varepsilon}_{\rm r} = \frac{1}{k_{\rm o}\beta_{\rm o}}\boldsymbol{k}_{\rm r}\times(\boldsymbol{k}_{\rm r}\times\boldsymbol{p}). \tag{66}$$

It can be verified that the polarization vectors of the reflected fields are related to the polarization vectors of the corresponding incident fields as

$$\boldsymbol{\varepsilon}_{\rm r}' = \boldsymbol{\varepsilon}_{\parallel}' - \boldsymbol{\varepsilon}_{\perp}', \tag{67}$$

$$\boldsymbol{\varepsilon}_{\rm r} = \boldsymbol{\varepsilon}_{\perp} - \boldsymbol{\varepsilon}_{\parallel}. \tag{68}$$

In the usual approach to reflection by and transmission through an interface, the cases of an *s*-polarized and a *p*-polarized incident field are considered separately. It is interesting to notice that with the present formalism no such distinction is necessary. The result for any polarization is covered by a single derivation.

8. The Sommerfeld half-plane

An exactly solvable case is the scattering of a plane wave by the Sommerfeld half-plane. The sheet S is the part $x > 0$ of the xy-plane, so the y-axis is the edge of the sheet. The scattered magnetic and electric fields can be found in closed form [11–13], and the sheet current density can be obtained by applying Equation (31) to the solution $\boldsymbol{B}(\boldsymbol{r})_{\rm sc}$. As an illustration, we consider an incident wave of the form (47) under normal incidence, so that $\boldsymbol{k}_{\rm o} = k_{\rm o}\boldsymbol{e}_z$. For the polarization of the electric field we take $\boldsymbol{\varepsilon} = \boldsymbol{e}_x$, and with Equation (56) this gives $\boldsymbol{\varepsilon}' = \boldsymbol{e}_y$ for the polarization of the magnetic field. Figure 3 shows the directions of the various vectors with respect to the half-plane. The value of the incident magnetic field in the plane $z=0$ follows from Equation (55) and is $\boldsymbol{B}(\boldsymbol{r})_{\rm inc} = (E/c)\boldsymbol{e}_y$. If the sheet were a mirror, the induced sheet current density would be

$$\boldsymbol{i}(\boldsymbol{r}) = 2\varepsilon_{\rm o}cE\boldsymbol{e}_x \quad \text{(mirror)}, \tag{69}$$

according to Equation (57). The current density is in the x-direction and has a constant complex amplitude. By solving Equations (42) and (43) simultaneously for the half-plane configuration, with methods similar to those applied in [13], it follows that the current density has the form

$$\boldsymbol{i}(\boldsymbol{r}) = 2\varepsilon_{\rm o}cE\eta(x)\boldsymbol{e}_x \tag{70}$$

and the function $\eta(x)$ is found to be

$$\eta(x) = 1 - \frac{F\big((k_{\rm o}x)^{1/2}\big)}{F(0)}, \tag{71}$$

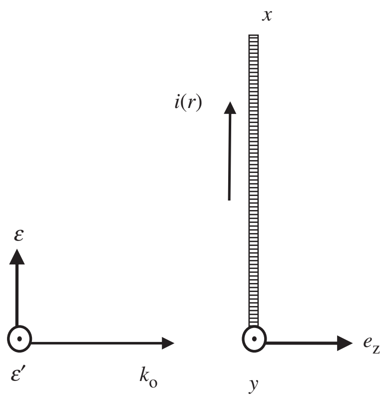

Figure 3. The figure illustrates the setup of the Sommerfeld half-plane example from Section 8. The half-plane is the part $x > 0$ of the xy-plane, and the z-axis is directed towards the shadow side of the sheet. The y-axis is the edge of the sheet. The incident plane wave is under normal incidence, as indicated by the wave vector $\boldsymbol{k}_o$, and the polarization vectors of the incident electric and magnetic fields are $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$, respectively.

with $F(\mu)$ the modified Fresnel integral, defined as

$$F(\mu) = \int_{\mu}^{\infty} \mathrm{d}t \exp(\mathrm{i}t^2). \tag{72}$$

The solution for the half-plane is similar in form as for a mirror, but the complex amplitude of the current density depends on x through the function $\eta(x)$. We see that $F(\mu \to \infty) = 0$, and therefore the function $\eta(x)$ approaches unity for x large. Then Equation (70) goes over into Equation (69) for the mirror. In other words, any deviation from $\eta(x) = 1$ is due to the presence of the edge along the y-axis. We also notice that $\eta(0) = 0$, so the sheet current density at the edge vanishes in amplitude.

The physical currents appear on the a- and b-sides of the sheet. If we write

$$\boldsymbol{i}_{\mathrm{a,\,b}}(\boldsymbol{r}) = 2\varepsilon_o c E \eta_{\mathrm{a,\,b}}(x)\boldsymbol{e}_x, \tag{73}$$

then we find from Equations (39) and (40)

$$\eta_{\mathrm{a}}(x) = 1 - \frac{F\big((k_o x)^{1/2}\big)}{2F(0)}, \tag{74}$$

$$\eta_{\mathrm{b}}(x) = -\frac{F\big((k_o x)^{1/2}\big)}{2F(0)} \tag{75}$$

and these functions are shown in Figure 4. Far from the edge we have $\eta_{\mathrm{a}}(x) \approx 1$ and $\eta_{\mathrm{b}}(x) \approx 0$, indicating that the current densities at the illuminated side and the shadow side approach their values for a mirror. An interesting feature of this solution is that at the edge we have

$$\eta_{\mathrm{a}}(0) = \frac{1}{2}, \quad \eta_{\mathrm{b}}(0) = -\frac{1}{2}, \tag{76}$$

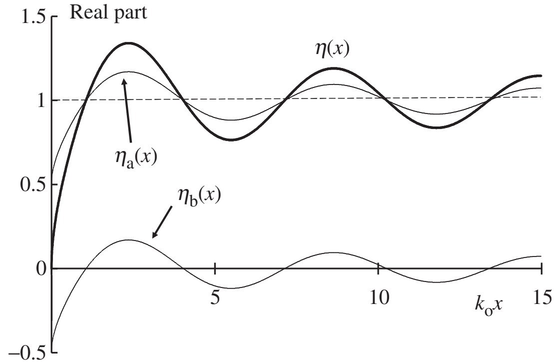

Figure 4. Shown are the real parts of the functions $\eta_a(x)$ and $\eta_b(x)$, which are proportional to the corresponding current densities in the Sommerfeld half-plane at $t = 0$. Also shown is their sum $\eta(x)$, representing the total sheet current. For a mirror with the same incident field we would have $\eta_a(x) = 1$ and $\eta_b(x) = 0$.

e.g. the current densities are finite and opposite in sign. The sheet current density is zero at the edge, but this is a consequence of the exact cancellation of the current densities at the two sides.

9. Conclusions

When light illuminates a flat thin perfectly-conducting sheet with edges, it induces surface current densities $\boldsymbol{i}_a(\boldsymbol{r})$ and $\boldsymbol{i}_b(\boldsymbol{r})$ at the lit and shadow sides of the material, respectively. Since the sheet is very thin, the effective sheet current density $\boldsymbol{i}(\boldsymbol{r}) = \boldsymbol{i}_a(\boldsymbol{r}) + \boldsymbol{i}_a(\boldsymbol{r})$ generates the scattered field. By applying the boundary conditions and an angular spectrum representation for the scattered field, we have shown that the difference between the current densities at both sides satisfies Equation (38), which has the incident magnetic field at the right-hand side. Therefore, if the sheet current density $\boldsymbol{i}(\boldsymbol{r})$ is known, the splitting in the separate surface current densities $\boldsymbol{i}_a(\boldsymbol{r})$ and $\boldsymbol{i}_b(\boldsymbol{r})$ is uniquely determined by the incident field. Equation (38) generalizes Equation (2) for a mirror, for which all current density is at the a-side, and Equations (39) and (40) give $\boldsymbol{i}_a(\boldsymbol{r})$ and $\boldsymbol{i}_b(\boldsymbol{r})$ explicitly in terms of the sheet current density and the incident magnetic field.

It was also shown that the Fourier transform of the sheet current density, $\boldsymbol{I}(\boldsymbol{k}_\parallel)$, must be a simultaneous solution of Equations (42) and (43). Once these equations are solved, the sheet current density $\boldsymbol{i}(\boldsymbol{r})$ can be found by inversion, as shown in Equation (32), after which the scattered magnetic and electric fields can be found, in principle, by integration. In Section 7 we have considered this approach for the simplest case of a plane wave incident upon a mirror. Equation (42) was solved explicitly, and it followed that the current densities and the reflected electric and magnetic fields could be obtained. In particular, the familiar splitting in *s*- and *p*-polarization could be avoided with the present formalism. In Section 8 we considered a plane wave incident upon the Sommerfeld half-plane, a case for which Equations (42) and (43) can be solved in closed form. Far away from the edge, the solution is approximately the solution for the mirror.

When the light is under normal incidence and polarized as depicted in Figure 3, it appeared that the current densities $\boldsymbol{i}_{\rm a}(\boldsymbol{r})$ and $\boldsymbol{i}_{\rm b}(\boldsymbol{r})$ at the edge are each others opposite, and therefore the sheet current density at the edge vanishes.

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