Current density in a perfect mirror

Henk F. Arnoldus

Department of Physics and Astronomy, Mississippi State University, P.O. Drawer 5167, Mississippi State, Mississippi 39762-5167, USA
e-mail: arnoldus@ra.msstate.edu

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Electromagnetic radiation incident upon a perfect mirror induces a current density on the surface of the conducting material of the mirror. It is shown that this surface current density can be expressed directly in terms of the source current density, which generates the incident field, without evaluating the electric and magnetic fields first. © 2008 Optical Society of America

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When electromagnetic radiation is incident upon a perfectly conducting object, it induces a current density $\mathbf{i}(\mathbf{r},t)$ on its surface. This surface current density generates electromagnetic radiation, and for a point outside the object this is observed as the reflected field. For a point inside the material the field generated by the surface current density exactly cancels the incident field. A common approach to the scattering of light is to consider the general solution of Maxwell's equations outside the object, in a suitable mode expansion, and then impose the boundary conditions at the interface [1]. The boundary conditions for a perfect conductor are that the tangential component of the electric field and the normal component of the magnetic field vanish just outside the material. Once the reflected field is obtained, the surface current density can be found from [2]

$$\mathbf{i}(\mathbf{r}) = \frac{1}{\mu_o}\hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}), \tag{1}$$

where $\hat{\mathbf{n}}(\mathbf{r})$ is the unit normal at point $\mathbf{r}$ on the surface, directed from the material into the vacuum. We assume a harmonic time dependence with angular frequency ω,

$$\mathbf{i}(\mathbf{r},t) = \mathrm{Re}[\mathbf{i}(\mathbf{r})e^{-i\omega t}], \tag{2}$$

and similarly $\mathbf{B}(\mathbf{r})$ in Eq. (1) is the complex amplitude of the total magnetic field at $\mathbf{r}$, just outside the object. In an alternative approach, integral equations for the electric or magnetic field are considered, which have the incident field as inhomogeneous term [3]. After solving such an integral equation, the current density on the object follows again from Eq. (1). When the material is a perfect conductor, an integral equation for the current density $\mathbf{i}(\mathbf{r})$ can be derived [4], which is due to Maue [5] and is known as the magnetic field integral equation. It has the incident magnetic field at the surface as inhomogeneous term. By using this equation, the current density can be found without considering the solution for the magnetic field first. After solving for $\mathbf{i}(\mathbf{r})$, the magnetic and electric fields can be obtained by integration with the Green's function. This method has been applied numerically to the scattering of plane waves off objects of a complicated shape [6,7].

In this Letter the reflection by a flat perfect mirror is considered, but the incident field is allowed to be emitted by a source with an arbitrary volume current density $\mathbf{j}(\mathbf{r})$. The induced surface current density $\mathbf{i}(\mathbf{r},t)$ can then have intriguing field line patterns, including vortices, singular points, singular circles, and loops, even for very simple sources such as an electric or a magnetic dipole [8,9]. As schematically illustrated in Fig. 1, the surface of the mirror is taken as the xy plane, and the unit normal is $\mathbf{e}_z$, which is directed towards the side of the incident field. The magnetic field integral equation for a mirror can be solved, with the result [10] that

$$\mathbf{i}(\mathbf{r}) = \frac{2}{\mu_o}\mathbf{e}_z \times \mathbf{B}(\mathbf{r})_{\mathrm{inc}}. \tag{3}$$

As compared with Eq. (1), here the right-hand side involves the incident magnetic field, rather than the total magnetic field, so when the incident field is known, we immediately obtain the current density in the surface. It is worth noting that the current density at point $\mathbf{r}$ is determined by the incident field at the same point $\mathbf{r}$, regardless of the current density $\mathbf{i}(\mathbf{r})$ at other points on the surface. We now show that the surface current density in the mirror can be expressed explicitly in terms of the volume current den-

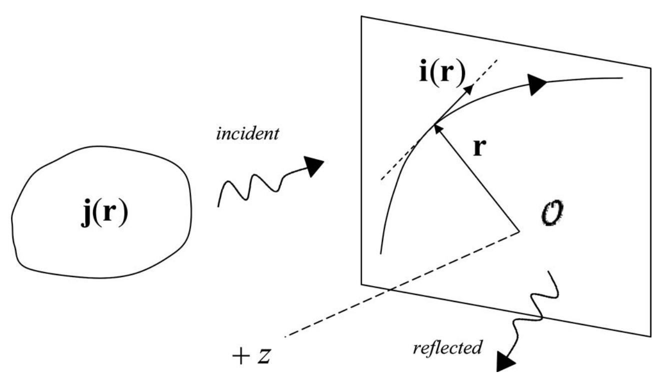

Fig. 1. A volume current density $\mathbf{j}(\mathbf{r})$ represents a source of radiation in front of a mirror. The surface of the mirror is the xy plane, and the z axis is directed towards the source. A surface current density $\mathbf{i}(\mathbf{r})$ is induced on the surface of the mirror, and this vector field can be represented by a pattern of field lines.

sity $\mathbf{j}(\mathbf{r})$ of the source. To this end, we recall that the incident magnetic field, emitted by the current source, is given by

$$\mathbf{B}(\mathbf{r})_{\mathrm{inc}} = \frac{\mu_o}{4\pi}\nabla \times \int \mathrm{d}^3\mathbf{r}' g(\mathbf{r}-\mathbf{r}')\mathbf{j}(\mathbf{r}'), \quad (4)$$

with $g(\mathbf{r})=\exp(ikr)/r$, $r=|\mathbf{r}|$, the free-space Green's function of the Helmholtz equation, and $k=\omega/c$. Combination of Eqs. (3) and (4) then yields

$$\mathbf{i}(\mathbf{r}) = \frac{1}{2\pi}\mathbf{e}_z \times \int \mathrm{d}^3\mathbf{r}' f(\mathbf{r}-\mathbf{r}')(\mathbf{r}-\mathbf{r}') \times \mathbf{j}(\mathbf{r}'), \quad (5)$$

where the function $f(\mathbf{r})$ is defined as

$$f(\mathbf{r}) = \frac{1}{r^2}\left[ik - \frac{1}{r}\right] e^{ikr}. \quad (6)$$

In Eq. (5), the integral runs over the source of the radiation, and the point $\mathbf{r}$ is a given point in the mirror. This shows that the surface current density $\mathbf{i}(\mathbf{r})$ at any point $\mathbf{r}$ of the mirror can be obtained directly from the given source current density $\mathbf{j}(\mathbf{r})$.

An object in front of a mirror has an image behind the mirror, and when the object is viewed as a current distribution, the image can be viewed as being emitted by a mirror-image current distribution behind the mirror [11]. The total electromagnetic field in front of the mirror is then the sum of the incident field and the field of the image source, and with Eq. (1) the surface current density in the mirror can be found. As an example we consider an electric dipole with dipole moment $\mathbf{d}(t)=\mathrm{Re}[\mathbf{d}\exp(-i\omega t)]$, in which the complex amplitude $\mathbf{d}$ is an arbitrary constant vector. When we write $\mathbf{d}=\mathbf{d}_\perp+\mathbf{d}_\parallel$, where the subscripts $\perp$ and $\parallel$ refer to the perpendicular and parallel parts of the vector with respect to the xy plane, then the image source is again an electric dipole, and it has $\mathbf{d}^{\mathrm{im}}=\mathbf{d}_\perp-\mathbf{d}_\parallel$ as its complex amplitude. The coordinates of the image dipole are $(x,y,-z)$, when (x,y,z) are the coordinates of the source dipole. With the known expression for the magnetic field emitted by an electric dipole, the surface current density can then be found from the boundary condition (1). On the other hand, when the dipole is located at position $\mathbf{r}_o$ in front of the mirror, as shown in Fig. 2, it has a current density

$$\mathbf{j}(\mathbf{r}) = -i\omega\mathbf{d}\,\delta(\mathbf{r}-\mathbf{r}_o), \quad (7)$$

and with Eq. (5) we immediately find

$$\mathbf{i}(\mathbf{r}) = \frac{i\omega}{2\pi} f(\mathbf{r}-\mathbf{r}_o)\mathbf{e}_z \times [\mathbf{d} \times (\mathbf{r}-\mathbf{r}_o)] \quad (8)$$

for the complex amplitude of the current density on the surface.

As an illustration, let us consider a dipole located on the z axis, at a distance H from the mirror, and with $\mathbf{d}=d_o\mathbf{e}_z, d_o>0$. Then the dipole moment oscillates harmonically along the z axis as $\mathbf{d}(t)=d_o\cos(\omega t)\mathbf{e}_z$, and with Eqs. (2) and (8) we obtain for the current density

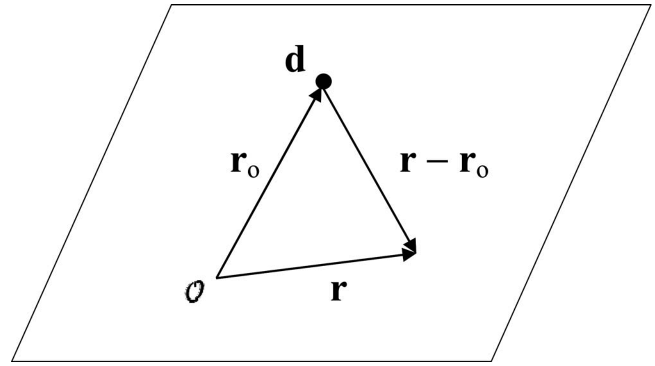

Fig. 2. An electric dipole $\mathbf{d}$ is located at position $\mathbf{r}_o$ in front of the mirror. The current density at point $\mathbf{r}$ of the surface depends on vector $\mathbf{r}-\mathbf{r}_o$, which is the field-point vector $\mathbf{r}$, relative to the position of the dipole.

$$\mathbf{i}(\mathbf{r},t) = \frac{ck^3 d_o}{2\pi}\frac{q}{q_1^2}\left[\cos(q_1-\omega t) - \frac{1}{q_1}\sin(q_1-\omega t)\right]\mathbf{e}_\rho. \quad (9)$$

Here we have introduced $q=kr$ for the dimensionless distance between the origin of coordinates and the field point $\mathbf{r}$ in the xy plane, and $q_1=k(r^2+H^2)^{1/2}$ represents the dimensionless distance between the dipole and the field point $\mathbf{r}$. The direction of $\mathbf{i}(\mathbf{r},t)$ is along the radial unit vector $\mathbf{e}_\rho$ in the xy plane, so the current is flowing radially outward or inward, depending on the sign of the expression in square brackets. For values of q for which $\tan(q_1-\omega t)=q_1$ the current density is zero. The solutions q of this transcendental equation correspond to circles in the xy plane, and across these circles the current density changes from flowing inward to flowing outward. Figure 3 shows the resulting field line pattern for the vector field $\mathbf{i}(\mathbf{r},t)$ in the xy plane for a given value of

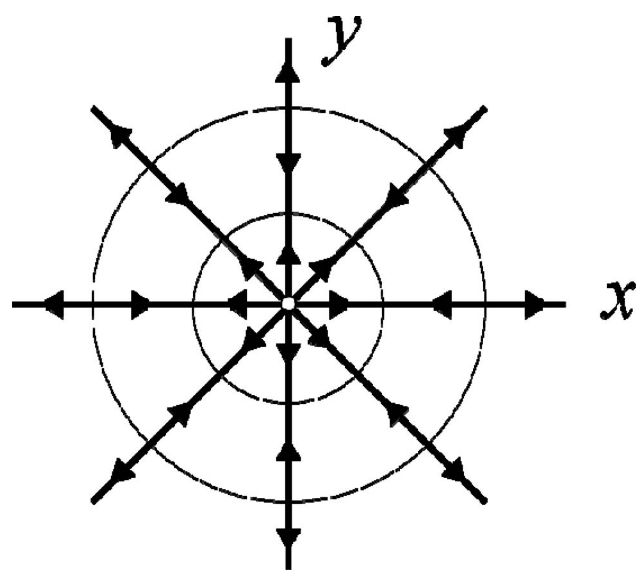

Fig. 3. Field lines of the current density in the mirror induced by a dipole oscillating along the z axis. The current is in the radial direction, and it reverses its orientation across each of the thin circles. There is an infinite set of these circles, which are spaced by about half an optical wavelength. As time progresses these circles expand, and new circles emanate from the origin of coordinates.

time t. When time progresses, the circles expand at near the speed of light, and the field line picture changes accordingly.

Equation (5) shows that the surface current density $\mathbf{i}(\mathbf{r})$ in the mirror can be obtained as an integral over the source region, involving the volume current density $\mathbf{j}(\mathbf{r})$ of the source. This surface current density can be evaluated without considering the radiation field explicitly, as would be necessary if $\mathbf{i}(\mathbf{r})$ had been computed from Eq. (1) or Eq. (3), and there is also no need to consider the image source. The method has been illustrated by the simple example of an electric dipole near the mirror, but the approach applies equally well to sources of a more complex structure.

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