Boundary conditions in an integral approach to scattering

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Scattering of electromagnetic radiation by an object of arbitrary shape or a structured surface, infinite in extent, is considered. When radiation is incident on an interface separating vacuum from a material medium, a current density is induced in the bulk and a surface current density may appear on the boundary surface. The electromagnetic field is then the sum of the incident field and the field generated by the current densities. This concept leads to expressions for the electric and magnetic fields that can easily be shown to be exact integrals of Maxwell's equations both in the vacuum and in the medium. At the boundary surface, the electric and magnetic fields must be discontinuous, with the discontinuity determined by the surface charge and current densities. This is usually referred to as boundary conditions for Maxwell's equations. We show that the integrals for the electric and magnetic fields automatically satisfy these boundary conditions, no matter the origin of the current densities. © 2006 Optical Society of America

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1. INTRODUCTION

Scattering of radiation by an obstacle and diffraction through an aperture in a screen are two of the most studied problems in electromagnetic theory. It has been realized for a long time that an integral approach to such problems may be more suitable than attempting to solve Maxwell's equations in combination with the appropriate boundary conditions. The earliest integral formulation seems to be Kirchhoff's integral theorem,[1–3] which has been applied widely to diffraction of radiation through an aperture. The theorem expresses a solution of the Helmholtz equation in a given point in terms of an integral over a closed surface surrounding that point, involving that same solution and its normal derivative at the surface. Evidently, one needs to make some approximations regarding the solution on the surface to arrive at an explicit result for the diffraction problem. Because of such approximations, the theory is not entirely consistent, and in particular the resulting solution usually violates the boundary conditions on the screen.[4] A different type of integral theorem is the Ewald–Oseen extinction theorem, and its various generalizations,[5–10] which has been applied to solve scattering problems.[11,12] The theorem involves again the solution of the field and its normal derivative on a boundary surface.

Both the Kirchhoff integral theorem and the extinction theorems follow from a combination of Maxwell's equations and some form of Green's theorem. In an entirely different approach, the electric field is first expressed as a sum of the incident field and the field generated by the current density in the material. For a linear medium, the unknown current density is proportional to the electric field, and this then leads to an integral equation for the electric field only. For the interesting case of a perfect conductor, the field inside the material vanishes, and all current is surface current. We then consider the magnetic field, and the requirement that this field is zero inside the perfect conductor leads to an equation for the surface current density only. After solving this equation, the electric and magnetic fields outside the perfect conductor can be obtained by integration. This method has been applied successfully in the study of scattering of electromagnetic radiation by dielectric or perfectly conducting objects of all sorts of shapes.[13–23]

We consider the situation shown in Fig. 1. Electromagnetic radiation is incident from vacuum upon an object of arbitrary shape. The boundary surface S may be closed, like for an isolated object, or it may extend to infinity, like for scattering off a half-space, or it may be the boundary of any combination of objects. In the material we have a current density $\mathbf{j}(\mathbf{r})$ and a charge density $\rho(\mathbf{r})$, which are related by the continuity equation

$$\rho(\mathbf{r}) = -\frac{i}{\omega}\nabla\cdot\mathbf{j}(\mathbf{r}). \tag{1}$$

We assume a harmonic time dependence throughout, so that $\rho(\mathbf{r},t)=\mathrm{Re}[\rho(\mathbf{r})\exp(-i\omega t)]$, and similarly for other time-dependent quantities. On the surface we may have a surface current density $\mathbf{i}(\mathbf{r})$ and surface charge density $\sigma(\mathbf{r})$. Conservation of charge at the surface can be expressed as[24]

$$\sigma(\mathbf{r}) = -\frac{i}{\omega}[\nabla_S\cdot\mathbf{i}(\mathbf{r}) - \hat{\mathbf{n}}(\mathbf{r})\cdot\mathbf{j}(\mathbf{r}_-)], \tag{2}$$

where $\nabla_S\cdot\mathbf{i}(\mathbf{r})$ is the surface divergence of $\mathbf{i}(\mathbf{r})$ (Appendix A), and $\hat{\mathbf{n}}(\mathbf{r})$ is the unit normal at point $\mathbf{r}$ on S, which is directed from the material to the vacuum. The field point $\mathbf{r}_-$ is just inside the medium, near $\mathbf{r}$ on S. We shall consider Eqs. (1) and (2) as the definitions of $\rho(\mathbf{r})$ and $\sigma(\mathbf{r})$, respectively, given $\mathbf{j}(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})$, so that conservation of charge is automatic.

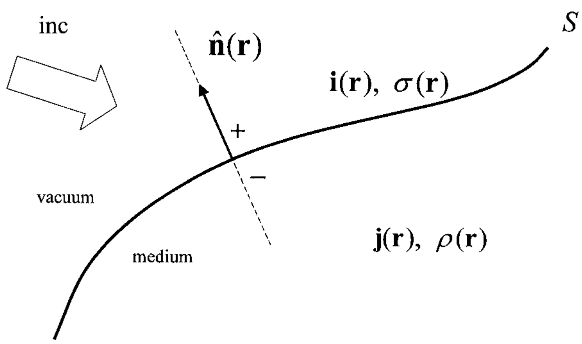

Fig. 1. Surface S is the boundary between vacuum and a material medium. An incident electromagnetic field induces a charge and current density in the material and on the surface. The unit normal $\hat{\mathbf{n}}(\mathbf{r})$ on S is directed from the medium to the vacuum.

The electric field $\mathbf{E}(\mathbf{r})$ and magnetic field $\mathbf{B}(\mathbf{r})$ must satisfy Maxwell's equations,

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_o}, \tag{3}$$

$$\nabla \times \mathbf{E}(\mathbf{r}) = i\omega \mathbf{B}(\mathbf{r}), \tag{4}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0, \tag{5}$$

$$\nabla \times \mathbf{B}(\mathbf{r}) = -\frac{i\omega}{c^2}\mathbf{E}(\mathbf{r}) + \mu_o \mathbf{j}(\mathbf{r}), \tag{6}$$

for any point $\mathbf{r}$ not on S. Inside the material, $\mathbf{j}(\mathbf{r})$ and $\rho(\mathbf{r})$ will have some value, and outside the material we have $\mathbf{j}(\mathbf{r})=0$, $\rho(\mathbf{r})=0$. For $\mathbf{r}$ on the boundary, the fields are discontinuous, and the boundary conditions can be written as

$$\mathbf{E}(\mathbf{r}_+) - \mathbf{E}(\mathbf{r}_-) = \frac{\sigma(\mathbf{r})}{\epsilon_o}\hat{\mathbf{n}}(\mathbf{r}), \tag{7}$$

$$\mathbf{B}(\mathbf{r}_+) - \mathbf{B}(\mathbf{r}_-) = \mu_o \mathbf{i}(\mathbf{r}) \times \hat{\mathbf{n}}(\mathbf{r}), \tag{8}$$

where $\mathbf{r}_+$ and $\mathbf{r}_-$ indicate field points just outside and inside the material, respectively, and near the point $\mathbf{r}$ on S. Equations (7) and (8) should be considered as representing Maxwell's equations for a point $\mathbf{r}$ on S. Usually, these equations are written as four equations, such as $\hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{E}(\mathbf{r}_+) = \hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{E}(\mathbf{r}_-)$, but for the present purpose it is more convenient to combine these as in Eqs. (7) and (8). When Maxwell's Eqs. (3)–(6) are solved, it is essential that the solutions satisfy the boundary conditions of Eqs. (7) and (8). We shall consider integrals of Maxwell's equations for $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$, in terms of given $\mathbf{j}(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})$, which satisfy Eqs. (3)–(6) at both sides of the surface, and we shall show that these solutions automatically satisfy the boundary conditions of Eqs. (7) and (8).

2. INTEGRALS OF MAXWELL'S EQUATIONS

Solutions to Eqs. (3)–(6) are most easily formulated in terms of Hertz vectors. For a given $\mathbf{j}(\mathbf{r})$ we define

$$\mathbf{\Pi}_\mathbf{j}(\mathbf{r}) = \frac{\mu_o}{4\pi}\int d^3\mathbf{r}'\mathbf{j}(\mathbf{r}')g(\mathbf{r}-\mathbf{r}'), \tag{9}$$

where

$$g(\mathbf{r}-\mathbf{r}') = \frac{e^{ik_o|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}, \tag{10}$$

with $k_o=\omega/c$, is the free-space Green's function for the scalar Helmholtz equation, and the integration in Eq. (9) runs over all material. We shall assume that a field point $\mathbf{r}$ is not exactly on the surface S. When the field point $\mathbf{r}$ is in the material, $g(\mathbf{r}-\mathbf{r}')$ has a singularity at $\mathbf{r}'=\mathbf{r}$. It is understood that a small sphere with radius δ around $\mathbf{r}$ is left out, and that eventually we let the radius of this sphere shrink to zero. Figure 2 illustrates this concept. In this way, the integral over the sphere adds to $\mathbf{\Pi}_\mathbf{j}(\mathbf{r})$, and this contribution vanishes for $\delta \to 0$, since

$$\frac{\mu_o}{4\pi}\mathbf{j}(\mathbf{r})\int_{\text{sphere}} d^3\mathbf{r}'g(\mathbf{r}-\mathbf{r}') \underset{\delta\to 0}{\to} 0, \tag{11}$$

as can be verified by integrating $g(\mathbf{r}-\mathbf{r}')$ over the sphere, and taking $\delta \to 0$. Green's function satisfies the equation $(\nabla^2+k_o^2)\, g(\mathbf{r}-\mathbf{r}')=0$ for $\mathbf{r}' \neq \mathbf{r}$. When considering $(\nabla^2+k_o^2)\,\mathbf{\Pi}_\mathbf{j}(\mathbf{r})$, this would give zero if we could just move the operator $\nabla^2+k_o^2$ under the integral sign. However, the exclusion of the small sphere gives an additional term, since the derivatives with respect to the Cartesian components of $\mathbf{r}$ move the location of the sphere. This is similar to the Leibniz rule for differentiation of integrals, for the case where the limits of integration depend on the differentiation variable. Taking this into account, we obtain[25,26]

$$(\nabla^2+k_o^2)\mathbf{\Pi}_\mathbf{j}(\mathbf{r}) = -\mu_o \mathbf{j}(\mathbf{r}). \tag{12}$$

Similarly we define the Hertz vector for the surface current density $\mathbf{i}(\mathbf{r})$ as

$$\mathbf{\Pi}_\mathbf{i}(\mathbf{r}) = \frac{\mu_o}{4\pi}\int dS'\mathbf{i}(\mathbf{r}')g(\mathbf{r}-\mathbf{r}'), \tag{13}$$

and this integral runs over all surfaces of the material. Since we assume that the field point $\mathbf{r}$ is not in S, the integrand in Eq. (13) has no singularity. Therefore here we can move $\nabla^2+k_o^2$ under the integral sign, so that

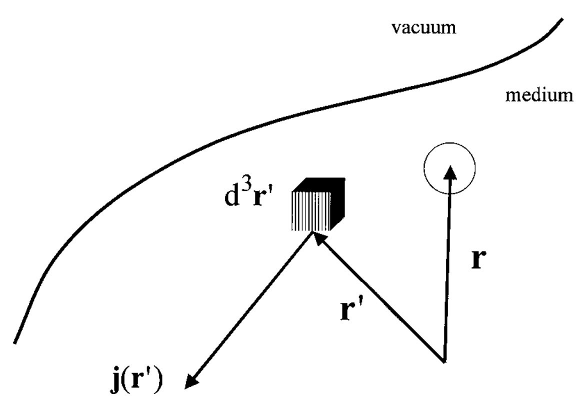

Fig. 2. For volume integrals involving Green's function $g(\mathbf{r}-\mathbf{r}')$, a small sphere around the singular point $\mathbf{r}'=\mathbf{r}$ is left out, and in the end we let the sphere shrink to a point.

$$(\nabla^2 + k_o^2)\mathbf{\Pi_i}(\mathbf{r}) = 0. \tag{14}$$

Let $\mathbf{E}(\mathbf{r})_{\text{inc}}$ and $\mathbf{B}(\mathbf{r})_{\text{inc}}$ represent the electric and magnetic incident fields, respectively, which are solutions of the homogeneous Maxwell's equations [Eqs. (3)–(6) with $\rho \equiv 0, \mathbf{j} \equiv 0$]. This incident electromagnetic field exists both in vacuum and inside the material. Inside the material this is the field that would be present if there were no material. The (total) field in the material may induce a charge density $\rho(\mathbf{r})$ and current density $\mathbf{j}(\mathbf{r})$, and in addition a surface charge density $\sigma(\mathbf{r})$ and surface current density $\mathbf{i}(\mathbf{r})$ may appear. The charge and current densities in turn generate an electromagnetic field, which adds to the incident field. It can then be checked by inspection that

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}(\mathbf{r})_{\text{inc}} + i\omega\langle \mathbf{\Pi_j}(\mathbf{r}) + \mathbf{\Pi_i}(\mathbf{r}) + \frac{1}{k_o^2}\nabla\{\nabla\cdot[\mathbf{\Pi_j}(\mathbf{r}) + \mathbf{\Pi_i}(\mathbf{r})]\}\rangle, \tag{15}$$

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}(\mathbf{r})_{\text{inc}} + \nabla\times[\mathbf{\Pi_j}(\mathbf{r}) + \mathbf{\Pi_i}(\mathbf{r})], \tag{16}$$

are solutions of Maxwell's Eqs. (3)–(6), both in vacuum and in the material. For the verification we only need Eqs. (12) and (14), some vector identities, and the definition of Eq. (1) of the charge density. It is interesting to note that the solution depends on $\mathbf{j}(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})$, through $\mathbf{\Pi_j}(\mathbf{r})$ and $\mathbf{\Pi_i}(\mathbf{r})$, but not explicitly on $\rho(\mathbf{r})$ and $\sigma(\mathbf{r})$. With a vector identity and Eqs. (12) and (14), the result for $\mathbf{E}(\mathbf{r})$ can also be written as

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}(\mathbf{r})_{\text{inc}} - \frac{i}{\epsilon_o\omega}\mathbf{j}(\mathbf{r}) + \frac{ic^2}{\omega}\nabla\times\{\nabla\times[\mathbf{\Pi_j}(\mathbf{r}) + \mathbf{\Pi_i}(\mathbf{r})]\}, \tag{17}$$

which we shall use in Section 6.

The solution of Eqs. (15) and (16) is the sum of the incident field and the field generated by the current densities. Outside the material, this is just the incident field plus the scattered field. Inside the material, however, the total field will have a different appearance. For instance, in a dielectric the wavenumber is $k = nk_o$, with n the index of refraction. Therefore, the incident field must be canceled exactly by the field generated by the current densities, a situation reminiscent of the extinction theorem. We also note that for the solution given by Eqs. (15) and (16), no assumption is made about the nature of the current densities, so these integrals of Maxwell's equations hold under any circumstance.

In Sections 3 and 4 we shall prove that the boundary conditions of Eqs. (7) and (8) are also satisfied for these solutions.

3. BOUNDARY CONDITION FOR THE MAGNETIC FIELD

The surface current density $\mathbf{i}(\mathbf{r})$ enters the solution given by Eqs. (15) and (16) through $\mathbf{\Pi_i}(\mathbf{r})$, and for the verification of this solution we only needed that $(\nabla^2 + k_o^2)\mathbf{\Pi_i}(\mathbf{r}) = 0$. Since this identity follows from the general form of the definition of $\mathbf{\Pi_i}(\mathbf{r})$, Eq. (13), we see that Eqs. (15) and (16) provide a solution of Maxwell's equations for any $\mathbf{i}(\mathbf{r})$. For instance, if we would simply leave out $\mathbf{\Pi_i}(\mathbf{r})$, then the solution would still satisfy Maxwell's equations for any $\mathbf{r}$ off S. The surface current density $\mathbf{i}(\mathbf{r})$ enters the problem through the boundary condition of Eq. (8), and we shall now show that for the magnetic field given by Eq. (16), this boundary condition is satisfied. Since $\mathbf{E}(\mathbf{r})$ is determined by $\mathbf{B}(\mathbf{r})$ through Maxwell's Eq. (6), this then determines the dependence of $\mathbf{E}(\mathbf{r})$ on $\mathbf{i}(\mathbf{r})$, with the result given by Eq. (15).

To prove that the solution of Eq. (16) satisfies the boundary condition of Eq. (8), we must show that the right-hand side of Eq. (16) jumps with $\mu_o\mathbf{i}(\mathbf{r})\times\hat{\mathbf{n}}(\mathbf{r})$ when we cross S at $\mathbf{r}$ going from the material to vacuum. To this end, we first note that $\mathbf{B}(\mathbf{r})_{\text{inc}}$ is continuous across S. The second term on the right-hand side of Eq. (16) is $\nabla\times\mathbf{\Pi_j}(\mathbf{r})$. The curl operator can be moved under the integral sign without an additional term appearing,[25] and with a vector identity we then have

$$\nabla\times\mathbf{\Pi_j}(\mathbf{r}) = -\frac{\mu_o}{4\pi}\int \mathrm{d}^3\mathbf{r}'\mathbf{j}(\mathbf{r}')\times\nabla g(\mathbf{r}-\mathbf{r}'). \tag{18}$$

Again, in this integral a small sphere around $\mathbf{r}$ is left out. In the integrand we now have

$$\nabla g(\mathbf{r}-\mathbf{r}') = (\mathbf{r}-\mathbf{r}')\left(\frac{ik_o}{|\mathbf{r}-\mathbf{r}'|^2} - \frac{1}{|\mathbf{r}-\mathbf{r}'|^3}\right)e^{ik_o|\mathbf{r}-\mathbf{r}'|}. \tag{19}$$

When we integrate $\nabla g(\mathbf{r}-\mathbf{r}')$ over a small sphere around $\mathbf{r}$, we find that the integral vanishes for $\delta\to 0$, as in relation (11), and therefore there is no finite contribution from this singularity. When we move the field point $\mathbf{r}$ across S, the only jump in $\nabla\times\mathbf{\Pi_j}(\mathbf{r})$ could possibly come from the contribution of the small sphere, and thus we conclude that $\nabla\times\mathbf{\Pi_j}(\mathbf{r})$ is continuous across S.

The third term on the right-hand side of Eq. (16) is $\nabla\times\mathbf{\Pi_i}(\mathbf{r})$, with $\mathbf{\Pi_i}(\mathbf{r})$ given by Eq. (13). The field point $\mathbf{r}$ is off S, and the integration runs over $\mathbf{r}'$ in S, so the integrand has no singularity, and we have

$$\nabla\times\mathbf{\Pi_i}(\mathbf{r}) = -\frac{\mu_o}{4\pi}\int \mathrm{d}S'\mathbf{i}(\mathbf{r}')\times\nabla g(\mathbf{r}-\mathbf{r}'). \tag{20}$$

Let us now consider field points $\mathbf{r}_\pm$, just off S, and near $\mathbf{r}$ in S, as in the boundary conditions. To be specific, let

$$\mathbf{r}_\pm = \mathbf{r} \pm \epsilon\hat{\mathbf{n}}(\mathbf{r}), \quad \epsilon > 0, \tag{21}$$

as illustrated in Fig. 3. With $\mathbf{r}$ now being a point on S, Eq. (20) should be written as

$$(\nabla\times\mathbf{\Pi_i})(\mathbf{r}_\pm) = -\frac{\mu_o}{4\pi}\int \mathrm{d}S'\mathbf{i}(\mathbf{r}')\times\nabla_\pm g(\mathbf{r}_\pm-\mathbf{r}'). \tag{22}$$

Then $\nabla_\pm g(\mathbf{r}_\pm-\mathbf{r}')$ is given by Eq. (19) with $\mathbf{r}$ replaced by $\mathbf{r}_\pm$. The singularity is then located at $\mathbf{r}'=\mathbf{r}_\pm$, which is just off S, but for $\epsilon\to 0$ the singular point approaches $\mathbf{r}$ in S. Just as we introduced the small sphere for the volume integrals, we now consider a circle of radius δ around $\mathbf{r}$ in the local tangent plane of S, as shown in Fig. 4. The integral in Eq. (22) then has a contribution from this circle, and from the remainder of S. For the integral over the small circle we have

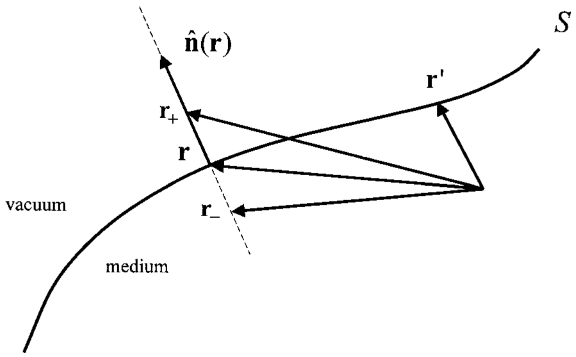

Fig. 3. For the verification of the boundary conditions, we let a field point approach the point $\mathbf{r}$ on the surface both from the inside ($\mathbf{r}_-$) and from the outside ($\mathbf{r}_+$). The integration variable is $\mathbf{r}'$.

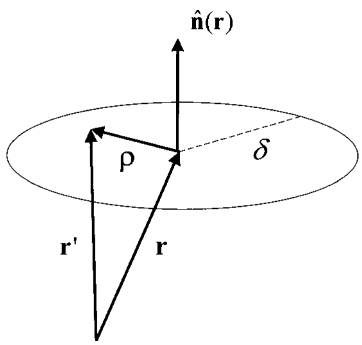

Fig. 4. Point $\mathbf{r}$ is in S, and $\mathbf{r}'$ is the integration variable for the surface integral of Eq. (22). Green's function $g(\mathbf{r}_\pm - \mathbf{r}')$ has a singularity at $\mathbf{r}_\pm$, so when $\mathbf{r}_-$ or $\mathbf{r}_+$ approaches the surface, this singularity approaches the point $\mathbf{r}$. To evaluate the limit $\mathbf{r}_\pm \to \mathbf{r}$, we leave out a small circle with radius δ around $\mathbf{r}$, and in the end we let $\delta \to 0$.

$$\int_{\text{circle}} \mathrm{d}S' \mathbf{i}(\mathbf{r}') \times \nabla_\pm g(\mathbf{r}_\pm - \mathbf{r}') \approx \mathbf{i}(\mathbf{r}) \times \int_{\text{circle}} \mathrm{d}S' \nabla_\pm g(\mathbf{r}_\pm - \mathbf{r}'). \quad (23)$$

To evaluate the integral on the right-hand side, we set $\boldsymbol{\rho} = \mathbf{r}' - \mathbf{r}$, which is a vector in the plane of the circle. Then the vector $\mathbf{r}_\pm - \mathbf{r}'$, which appears in $\nabla_\pm g(\mathbf{r}_\pm - \mathbf{r}')$, becomes

$$\mathbf{r}_\pm - \mathbf{r}' = \pm \epsilon \hat{\mathbf{n}}(\mathbf{r}) - \boldsymbol{\rho}, \quad (24)$$

and we have

$$\nabla_\pm g(\mathbf{r}_\pm - \mathbf{r}') = (\pm \epsilon \hat{\mathbf{n}}(\mathbf{r}) - \boldsymbol{\rho}) \left[\frac{ik_\mathrm{o}}{\rho^2 + \epsilon^2} - \frac{1}{(\rho^2 + \epsilon^2)^{3/2}} \right] e^{ik_\mathrm{o}\sqrt{\rho^2+\epsilon^2}}, \quad (25)$$

with ρ the magnitude of $\boldsymbol{\rho}$. The term proportional to $\boldsymbol{\rho}$ integrates to zero over the circle by symmetry. For the term proportional to ϵ we set $x = \rho^2 + \epsilon^2$, which yields

$$\int_{\text{circle}} \mathrm{d}S' \nabla_\pm g(\mathbf{r}_\pm - \mathbf{r}') = \pm \pi \epsilon \hat{\mathbf{n}}(\mathbf{r}) \int_{\epsilon^2}^{\delta^2+\epsilon^2} \mathrm{d}x \left(\frac{ik_\mathrm{o}}{x} - \frac{1}{x^{3/2}} \right) e^{ik_\mathrm{o}\sqrt{x}}. \quad (26)$$

For $\epsilon \to 0$, $\delta \to 0$, the integration variable x is small, and we can expand $\exp(ik_\mathrm{o}\sqrt{x})$ in a Taylor series around $x=0$. Keeping only the terms that may remain finite for $\epsilon \to 0$, $\delta \to 0$, we then obtain

$$\int_{\text{circle}} \mathrm{d}S' \nabla_\pm g(\mathbf{r}_\pm - \mathbf{r}') = \pm 2\pi \hat{\mathbf{n}}(\mathbf{r}) \left(\frac{\epsilon}{\sqrt{\delta^2 + \epsilon^2}} - 1 \right). \quad (27)$$

We now let $\epsilon \to 0$ first, corresponding to $\mathbf{r}_\pm \to \mathbf{r}$, which gives

$$\int_{\text{circle}} \mathrm{d}S' \nabla_\pm g(\mathbf{r}_\pm - \mathbf{r}') = \mp 2\pi \hat{\mathbf{n}}(\mathbf{r}), \quad (28)$$

and this is finite for $\delta \to 0$. For the integral over the remainder of S we simply set $\mathbf{r}_\pm = \mathbf{r}$, since there is no singularity in the integrand in Eq. (22) for $\epsilon \to 0$. Then in this integral we contract the circle. There is no contribution to this integral from the circle, since this gives the same contribution as the term proportional to $\boldsymbol{\rho}$ in the integral over the circle, which was found to be zero. In this sense, this integral becomes a principal value integral. We therefore find

$$\int \mathrm{d}S' \mathbf{i}(\mathbf{r}') \times \nabla_\pm g(\mathbf{r}_\pm - \mathbf{r}') = \mp 2\pi \mathbf{i}(\mathbf{r}) \times \hat{\mathbf{n}}(\mathbf{r}) + P \int \mathrm{d}S' \mathbf{i}(\mathbf{r}') \times \nabla g(\mathbf{r} - \mathbf{r}'), \quad (29)$$

where in the principal value integral the point $\mathbf{r}$ is in S. This integral is the same for the plus and minus solution, so with Eq. (22) we find for the difference

$$(\nabla \times \boldsymbol{\Pi}_\mathbf{i})(\mathbf{r}_+) - (\nabla \times \boldsymbol{\Pi}_\mathbf{i})(\mathbf{r}_-) = \mu_\mathrm{o} \mathbf{i}(\mathbf{r}) \times \hat{\mathbf{n}}(\mathbf{r}), \quad (30)$$

and from Eq. (16) we then obtain for the jump in the magnetic field $\mathbf{B}(\mathbf{r}_+) - \mathbf{B}(\mathbf{r}_-) = \mu_\mathrm{o} \mathbf{i}(\mathbf{r}) \times \hat{\mathbf{n}}(\mathbf{r})$, which is the boundary condition of Eq. (8).

4. BOUNDARY CONDITION FOR THE ELECTRIC FIELD

We now consider the change in $\mathbf{E}(\mathbf{r})$, given by Eq. (15), when the field point $\mathbf{r}$ crosses S. The incident field $\mathbf{E}(\mathbf{r})_{\mathrm{inc}}$ is continuous across S, and $\boldsymbol{\Pi}_\mathbf{j}(\mathbf{r})$ is continuous for the same reasons as discussed following Eq. (18). Therefore, the discontinuity in $\mathbf{E}(\mathbf{r})$ across S has to come from the term with $\nabla\{\nabla \cdot [\boldsymbol{\Pi}_\mathbf{j}(\mathbf{r}) + \boldsymbol{\Pi}_\mathbf{i}(\mathbf{r})]\}$. The $\nabla \cdot \boldsymbol{\Pi}_\mathbf{j}(\mathbf{r})$ is continuous across S following the same arguments as for $\nabla \times \boldsymbol{\Pi}_\mathbf{j}(\mathbf{r})$ discussed following Eq. (18). For the $\nabla \cdot \boldsymbol{\Pi}_\mathbf{i}(\mathbf{r})$ we repeat the same calculation as for $\nabla \times \boldsymbol{\Pi}_\mathbf{i}(\mathbf{r})$ following Eq. (20). The result is again Eq. (30), with $\times$ replaced by $\cdot$, and the right-hand side gets an additional minus sign. But since $\mathbf{i}(\mathbf{r})$ is in the tangent plane, we have $\mathbf{i}(\mathbf{r}) \cdot \hat{\mathbf{n}}(\mathbf{r}) = 0$, so that the $\nabla \cdot \boldsymbol{\Pi}_\mathbf{i}(\mathbf{r})$ is continuous.

With some vector identities and $\nabla g(\mathbf{r} - \mathbf{r}') = -\nabla' g(\mathbf{r} - \mathbf{r}')$, we have

$$\nabla \cdot \mathbf{\Pi_j}(\mathbf{r}) = \frac{i\omega}{c^2}\Pi_\rho(\mathbf{r}) - \frac{\mu_o}{4\pi}\int \mathrm{d}^3\mathbf{r}'\nabla' \cdot [\mathbf{j}(\mathbf{r}')g(\mathbf{r}-\mathbf{r}')]. \tag{31}$$

Here we have eliminated $\nabla \cdot \mathbf{j}(\mathbf{r})$ in favor of $\rho(\mathbf{r})$, Eq. (1), and introduced

$$\Pi_\rho(\mathbf{r}) = \frac{1}{4\pi\epsilon_o}\int \mathrm{d}^3\mathbf{r}'\rho(\mathbf{r}')g(\mathbf{r}-\mathbf{r}'). \tag{32}$$

When we take the gradient of Eq. (31), then $\nabla\Pi_\rho(\mathbf{r})$ is continuous across S, with the same arguments following Eq. (18). For the integral on the right-hand side of Eq. (31) we consider a field point $\mathbf{r}$ close to S, either inside or outside the material, like $\mathbf{r}_\pm$ in Section 3. We then divide the material in a volume $\bar{V}$ near $\mathbf{r}$, and the remainder of the material, as shown schematically in Fig. 5. The volume $\bar{V}$ does not have to be small. When $\mathbf{r}$ is inside the material, we leave out a small sphere around $\mathbf{r}$, as in Fig. 2. The integral over the remainder of the material is continuous when we move $\mathbf{r}$ across S. With the divergence theorem we then have for the contribution from $\bar{V}$

$$\int_{\bar{V}} \mathrm{d}^3\mathbf{r}'\nabla' \cdot [\mathbf{j}(\mathbf{r}')g(\mathbf{r}-\mathbf{r}')] = \int \mathrm{d}A' g(\mathbf{r}-\mathbf{r}')\mathbf{j}(\mathbf{r}') \cdot \hat{\mathbf{N}}(\mathbf{r}'), \tag{33}$$

with $\hat{\mathbf{N}}(\mathbf{r}')$ the outward unit normal on the surface of $\bar{V}$. For $\mathbf{r}$ inside the material, the right-hand side also includes the surface integral over the small sphere, but this integral is zero, as can be shown by explicit calculation or as seen from symmetry. The surface of $\bar{V}$ has a part $\bar{S}$, which coincides with S, and any discontinuity can only come from the contribution of the integral over $\bar{S}$. On $\bar{S}$ we have $\hat{\mathbf{N}}=\hat{\mathbf{n}}$ and $\mathrm{d}A'=\mathrm{d}S'$. Taking the gradient of Eq. (33) then gives

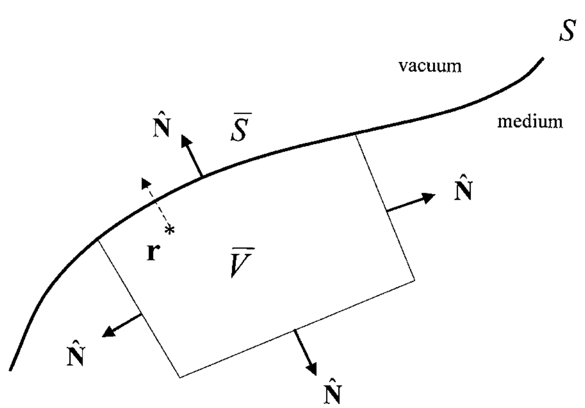

Fig. 5. For the evaluation of the discontinuity of the electric field across S we consider a volume $\bar{V}$ of the material. Part of its surface is $\bar{S}$, which coincides with S. Point $\mathbf{r}$ is near S, and we determine the change in the electric field when $\mathbf{r}$ crosses S.

$$\nabla\int_{\bar{V}} \mathrm{d}^3\mathbf{r}'\nabla' \cdot [\mathbf{j}(\mathbf{r}')g(\mathbf{r}-\mathbf{r}')] = \int_{\bar{S}} \mathrm{d}S'[\mathbf{j}(\mathbf{r}') \cdot \hat{\mathbf{n}}(\mathbf{r}')]\nabla g(\mathbf{r}-\mathbf{r}') + \cdots. \tag{34}$$

The integral on the right-hand side has the same appearance as the integral in Eq. (20). We leave out the $\times$ and replace $\mathbf{i}(\mathbf{r}')$ by $\mathbf{j}(\mathbf{r}')\cdot\hat{\mathbf{n}}(\mathbf{r}')$. The discontinuity of this integral when $\mathbf{r}$ crosses S is given by Eq. (29), so here the jump is $(\ldots)_+-(\ldots)_-=-4\pi[\mathbf{j}(\mathbf{r}_-)\cdot\hat{\mathbf{n}}(\mathbf{r})]\hat{\mathbf{n}}(\mathbf{r})$. We write $\mathbf{j}(\mathbf{r}_-)$, since this is the current density at the surface, just inside the material. With Eq. (31) we then obtain for the jump in $\nabla[\nabla\cdot\mathbf{\Pi_j}(\mathbf{r})]$

$$[\nabla(\nabla \cdot \mathbf{\Pi_j})](\mathbf{r}_+) - [\nabla(\nabla \cdot \mathbf{\Pi_j})](\mathbf{r}_-) = \mu_o[\mathbf{j}(\mathbf{r}_-) \cdot \hat{\mathbf{n}}(\mathbf{r})]\hat{\mathbf{n}}(\mathbf{r}). \tag{35}$$

From Eq. (13) we have

$$\nabla \cdot \mathbf{\Pi_i}(\mathbf{r}) = -\frac{\mu_o}{4\pi}\int \mathrm{d}S'\mathbf{i}(\mathbf{r}') \cdot \nabla' g(\mathbf{r}-\mathbf{r}'), \tag{36}$$

and with Eq. (A12) from Appendix A this is

$$\nabla \cdot \mathbf{\Pi_i}(\mathbf{r}) = \frac{\mu_o}{4\pi}\int \mathrm{d}S' g(\mathbf{r}-\mathbf{r}')\nabla'_S \cdot \mathbf{i}(\mathbf{r}') - \frac{\mu_o}{4\pi}\int \mathrm{d}S'\nabla'_S \cdot [\mathbf{i}(\mathbf{r}')g(\mathbf{r}-\mathbf{r}')], \tag{37}$$

in analogy to Eq. (31) for $\nabla\cdot\mathbf{\Pi_j}(\mathbf{r})$. Since we now have surface integrals, we divide S in $\bar{S}$ and the remainder of S, as in Fig. 5. Then any jump can only come from the integral over $\bar{S}$. The second integral on the right-hand side of Eq. (37) can be written as a loop integral over the curve bounding $\bar{S}$ by means of the surface analog of the divergence theorem. The singularity in the integrand is located at point $\mathbf{r}$, so when integrating around the loop, one never gets close to the singularity. Therefore, this integral and its gradient are continuous when we move $\mathbf{r}$ across S. The contribution from the first integral in Eq. (37) is

$$\nabla[\nabla \cdot \mathbf{\Pi_i}(\mathbf{r})] = \frac{\mu_o}{4\pi}\int \mathrm{d}S'[\nabla'_S \cdot \mathbf{i}(\mathbf{r}')]\nabla g(\mathbf{r}-\mathbf{r}') + \ldots, \tag{38}$$

and this integral is again of the form of the integral in Eq. (20). Therefore, the discontinuity in $\nabla[\nabla\cdot\mathbf{\Pi_i}(\mathbf{r})]$ is

$$[\nabla(\nabla \cdot \mathbf{\Pi_i})](\mathbf{r}_+) - [\nabla(\nabla \cdot \mathbf{\Pi_i})](\mathbf{r}_-) = -\mu_o[\nabla_S \cdot \mathbf{i}(\mathbf{r})]\hat{\mathbf{n}}(\mathbf{r}). \tag{39}$$

It is interesting to see that the discontinuity here comes from the first term in Eq. (37), whereas the jump shown in Eq. (35) comes from the second term in Eq. (31).

Now we add Eqs. (35) and (39):

$$\{\nabla[\nabla \cdot (\mathbf{\Pi_j}+\mathbf{\Pi_i})]\}(\mathbf{r}_+) - \{\nabla[\nabla \cdot (\mathbf{\Pi_j}+\mathbf{\Pi_i})]\}(\mathbf{r}_-) = \mu_o[\mathbf{j}(\mathbf{r}_-) \cdot \hat{\mathbf{n}}(\mathbf{r}) - \nabla_S \cdot \mathbf{i}(\mathbf{r})]\hat{\mathbf{n}}(\mathbf{r}), \tag{40}$$

and with Eq. (2) this becomes

$$\{\nabla[\nabla\cdot(\mathbf{\Pi_j}+\mathbf{\Pi_i})]\}(\mathbf{r}_+)-\{\nabla[\nabla\cdot(\mathbf{\Pi_j}+\mathbf{\Pi_i})]\}(\mathbf{r}_-)$$

$$=-i\omega\mu_o\sigma(\mathbf{r})\hat{\mathbf{n}}(\mathbf{r}). \qquad (41)$$

When we multiply this by $i\omega/k_o^2$, as in Eq. (15), we obtain $\mathbf{E}(\mathbf{r}_+)-\mathbf{E}(\mathbf{r}_-)=\sigma(\mathbf{r})\hat{\mathbf{n}}(\mathbf{r})/\epsilon_0$, which is the boundary condition of Eq. (7) for the electric field.

5. SCATTERING OFF A SINGLE OBJECT

Equations (15) and (16) give the solutions $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ of Maxwell's equations. We have shown that these solutions satisfy Maxwell's equations both in vacuum and in the material, and that the boundary conditions of Eqs. (7) and (8) are automatically satisfied by these expressions. The solutions are expressed in terms of $\mathbf{\Pi_j}(\mathbf{r})$ and $\mathbf{\Pi_i}(\mathbf{r})$, which are determined by $\mathbf{j}(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})$, respectively. There are no restrictions on the shape of the vacuum–material interface. It could be infinite in extent, like in scattering off a semi-infinite substrate, or the interface may consist of multiple surfaces, like in scattering off a collection of particles. In this section we consider scattering off a single object of finite size. In that case there is a single interface, and the surface S is a closed surface. For this situation, the solution for $\mathbf{E}(\mathbf{r})$, Eq. (15), can be simplified. In Eq. (31) the volume integral runs over the volume of the object, and since its surface is now a closed surface, the divergence theorem applies. Equation (31) then becomes

$$\nabla\cdot\mathbf{\Pi_j}(\mathbf{r})=\frac{i\omega}{c^2}\Pi_\rho(\mathbf{r})-\frac{\mu_o}{4\pi}\int \mathrm{d}S' g(\mathbf{r}-\mathbf{r}')\hat{\mathbf{n}}(\mathbf{r}')\cdot\mathbf{j}(\mathbf{r}'). \qquad (42)$$

There is also the integral over the small sphere that is excluded from the range of integration when the field point $\mathbf{r}$ is inside the material, but that surface integral vanishes. The $\mathbf{j}(\mathbf{r}')$ in the integrand is the current density just inside the surface, so in the notation of the boundary conditions, this is $\mathbf{j}(\mathbf{r}_-)$. Then, in Eq. (37) the second term on the right-hand side is the integral over a surface divergence, which equals a loop integral around the boundary of the surface with the surface divergence theorem. But for a closed surface this boundary curve shrinks to a point, and the integral is zero. Therefore, Eq. (37) becomes

$$\nabla\cdot\mathbf{\Pi_i}(\mathbf{r})=\frac{\mu_o}{4\pi}\int \mathrm{d}S' g(\mathbf{r}-\mathbf{r}')\nabla_S'\cdot\mathbf{i}(\mathbf{r}'). \qquad (43)$$

When we add Eqs. (42) and (43) the two integrals combine, and in the integrand we get the combination $\nabla_S'\cdot\mathbf{i}(\mathbf{r}')-\hat{\mathbf{n}}(\mathbf{r}')\cdot\mathbf{j}(\mathbf{r}')$. From Eq. (2) we see that this is just $i\omega\sigma(\mathbf{r}')$. Then we introduce

$$\Pi_\sigma(\mathbf{r})=\frac{1}{4\pi\epsilon_o}\int \mathrm{d}S'\sigma(\mathbf{r}')g(\mathbf{r}-\mathbf{r}'), \qquad (44)$$

after which we obtain for the sum of Eqs. (42) and (43):

$$\nabla\cdot[\mathbf{\Pi_j}(\mathbf{r})+\mathbf{\Pi_i}(\mathbf{r})]=\frac{i\omega}{c^2}[\Pi_\rho(\mathbf{r})+\Pi_\sigma(\mathbf{r})]. \qquad (45)$$

With this relation, Eq. (15) simplifies to

$$\mathbf{E}(\mathbf{r})=\mathbf{E}(\mathbf{r})_{\mathrm{inc}}+i\omega[\mathbf{\Pi_j}(\mathbf{r})+\mathbf{\Pi_i}(\mathbf{r})]-\nabla[\Pi_\rho(\mathbf{r})+\Pi_\sigma(\mathbf{r})], \qquad (46)$$

and Eq. (16) for $\mathbf{B}(\mathbf{r})$ remains the same.

An interesting question is under which condition Eq. (46) is a correct solution of Maxwell's equations, given $\mathbf{j}(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})$. Since Eq. (16) for $\mathbf{B}(\mathbf{r})$ is the same, the boundary condition for $\mathbf{B}(\mathbf{r})$ is still correct, and it is easy to see that the term $\nabla\Pi_\sigma(\mathbf{r})$ in Eq. (46) gives the correct jump in $\mathbf{E}(\mathbf{r})$ at the boundary. The $\Pi_\rho(\mathbf{r})$ and $\Pi_\sigma(\mathbf{r})$ satisfy the Helmholtz equations

$$(\nabla^2+k_o^2)\Pi_\rho(\mathbf{r})=-\frac{\rho(\mathbf{r})}{\epsilon_o}, \qquad (47)$$

$$(\nabla^2+k_o^2)\Pi_\rho(\mathbf{r})=0. \qquad (48)$$

When we take the divergence of Eq. (46) we obtain

$$\nabla\cdot\mathbf{E}(\mathbf{r})=\frac{\rho(\mathbf{r})}{\epsilon_o}+i\omega\{\nabla\cdot[\mathbf{\Pi_j}(\mathbf{r})+\mathbf{\Pi_i}(\mathbf{r})]-\frac{i\omega}{c^2}[\Pi_\rho(\mathbf{r})+\Pi_\sigma(\mathbf{r})]\}, \qquad (49)$$

and we see that, compared with Maxwell's Eq. (3), the additional term in braces appears. So for Eq. (46) to be a solution of Maxwell's equations, this term has to disappear. Setting this term equal to zero gives exactly Eq. (45). For Maxwell's Eq. (6) the same term appears, whereas Eqs. (4) and (5) still hold without further assumptions. Therefore, Eq. (46) provides a solution of Maxwell's equations under the condition that Eq. (45) holds. A sufficient condition is that the boundary interface is a single closed surface.

6. DIELECTRIC OR METALLIC MEDIUM

Equations (15) and (16) are solutions of Maxwell's equations, expressed in terms of given $\mathbf{j}(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})$. It is important to notice that nothing has been assumed about these current densities and also that no assumptions have been made about the material in which these current densities appear. Let us now consider the most common situation for which the material is a linear dielectric or a metal. For such materials the current density is proportional to the electric field according to

$$\mathbf{j}(\mathbf{r})=-i\omega\epsilon_o\chi(\mathbf{r})\mathbf{E}(\mathbf{r}), \qquad (50)$$

with $\chi(\mathbf{r})$ the complex-valued susceptibility function, which includes the conductivity. There is no surface current, so $\mathbf{\Pi_i}(\mathbf{r})=0$. With Eq. (50) for $\mathbf{j}(\mathbf{r})$ we obtain from Eqs. (9) and (17)

$$\epsilon(\mathbf{r})\mathbf{E}(\mathbf{r}) = \mathbf{E}(\mathbf{r})_{\text{inc}} + \frac{1}{4\pi}\nabla \times \left[\nabla \times \int \mathrm{d}^3\mathbf{r}'\chi(\mathbf{r}')g(\mathbf{r}-\mathbf{r}')\mathbf{E}(\mathbf{r}')\right], \tag{51}$$

with

$$\epsilon(\mathbf{r}) = 1 + \chi(\mathbf{r}). \tag{52}$$

Equation (51) is a linear integral equation for $\mathbf{E}(\mathbf{r})$ with $\mathbf{E}(\mathbf{r})_{\text{inc}}$ as the inhomogeneous term. The integral on the right-hand side only runs over the volume of the material, since $\chi(\mathbf{r})=0$ in vacuum. Equation (51) has been used frequently in scattering and diffraction problems. Numerically, one expands $\mathbf{E}(\mathbf{r})$ and $\mathbf{E}(\mathbf{r})_{\text{inc}}$ onto a set of discrete basis functions, which turns Eq. (51) into a set of linear equations for the expansion coefficients. This is known as the method of moments.[27] After solving Eq. (51), the magnetic field follows from Eq. (4). It should be noted that there exist many alternative integral equations for scattering and diffraction problems involving dielectric and metallic materials,[28] some of which may have computational advantage over Eq. (51).

7. PERFECT CONDUCTOR

A particularly interesting example is the case in which the material is a perfect conductor. In such a material the electric field is zero, and it then follows from Maxwell's equations that $\mathbf{B}$, ρ, and $\mathbf{j}$ also vanish. We only have surface charge and current densities as the source of radiation. We now consider the integral of Eq. (16) for the magnetic field, with $\mathbf{\Pi_j}(\mathbf{r})=0$. Inside the material we have $\mathbf{B}(\mathbf{r})=0$, so with Eq. (20) we find

$$\frac{\mu_o}{4\pi}\int \mathrm{d}S'\mathbf{i}(\mathbf{r}') \times \nabla g(\mathbf{r}-\mathbf{r}') = \mathbf{B}(\mathbf{r})_{\text{inc}}, \quad (\mathbf{r}\text{ in material}). \tag{53}$$

Given the incident field, this is an equation for the surface current density $\mathbf{i}(\mathbf{r})$. The incident field induces a surface current density, which is the solution of Eq. (53), and given this solution $\mathbf{i}(\mathbf{r})$, the magnetic field outside the material can be obtained from Eq. (16). The electric field outside the material then follows from Maxwell's Eq. (6). Equation (53) expresses that for $\mathbf{r}$ inside the material the field generated by $\mathbf{i}(\mathbf{r})$ cancels exactly the incident field, a situation reminiscent of the extinction theorem.

A more attractive form of Eq. (53) can be derived as follows. Equation (53) has to hold for all $\mathbf{r}$ in the material, so it holds in particular for points $\mathbf{r}_-$ just inside the material. For such points, the integral in Eq. (53) is the same as in Eq. (29), and we have

$$\frac{1}{2}\mu_o\mathbf{i}(\mathbf{r}) \times \hat{\mathbf{n}}(\mathbf{r}) + \frac{\mu_o}{4\pi}P\int \mathrm{d}S'\mathbf{i}(\mathbf{r}') \times \nabla g(\mathbf{r}-\mathbf{r}') = \mathbf{B}(\mathbf{r})_{\text{inc}}, \quad (\mathbf{r}\text{ in }S). \tag{54}$$

This is still an equation for $\mathbf{i}(\mathbf{r})$, but it now only involves points on the boundary surface S. Another simplification can be made by taking the cross product with $\hat{\mathbf{n}}(\mathbf{r})$:

$$\frac{1}{2}\mu_o\mathbf{i}(\mathbf{r}) + \frac{\mu_o}{4\pi}\hat{\mathbf{n}}(\mathbf{r}) \times P\int \mathrm{d}S'\mathbf{i}(\mathbf{r}') \times \nabla g(\mathbf{r}-\mathbf{r}') = \hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})_{\text{inc}}, \quad (\mathbf{r}\text{ in }S). \tag{55}$$

Here every term is in the local tangent plane, whereas in Eq. (54) both $\mathbf{B}(\mathbf{r})_{\text{inc}}$ and the integral have components normal to the surface. Therefore the formulation in Eq. (55) reduces the problem to a two-dimensional problem, although $\nabla g(\mathbf{r}-\mathbf{r}')$ is still three dimensional. The theorem was used in this form in Ref. 23.

Finally, let us consider the case where the surface is flat. In the integrand of the integral in Eq. (55), both the points $\mathbf{r}$ and $\mathbf{r}'$ are in S, and when S is flat, we have that the vector $\mathbf{r}-\mathbf{r}'$ lies in S. From Eq. (19) we see that $\nabla g(\mathbf{r}-\mathbf{r}')$ is proportional to $\mathbf{r}-\mathbf{r}'$, so this vector lies in S. Then $\mathbf{i}(\mathbf{r}') \times \nabla g(\mathbf{r}-\mathbf{r}')$ is perpendicular to S, and therefore the integral is a vector perpendicular to S. When we take the cross product with $\hat{\mathbf{n}}(\mathbf{r})$, this term vanishes, and we obtain

$$\mathbf{i}(\mathbf{r}) = \frac{2}{\mu_o}\hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})_{\text{inc}}. \tag{56}$$

We find the remarkable result that the induced current density at point $\mathbf{r}$ in S is determined by the value of the incident magnetic field at that same point. The corresponding surface charge density follows from Eq. (2). For a flat surface, temporarily taken to be the xy plane, the surface divergence is simply $\nabla_S \cdot \mathbf{i} = \partial i_x/\partial x + \partial i_y/\partial y$, and $\hat{\mathbf{n}}(\mathbf{r})=\mathbf{e}_z$. From Eq. (56) we then find

$$\nabla_S \cdot \mathbf{i} = -\frac{2}{\mu_o}[\nabla \times \mathbf{B}(\mathbf{r})_{\text{inc}}]_z. \tag{57}$$

From Maxwell's Eq. (6) we have $\nabla \times \mathbf{B}(\mathbf{r})_{\text{inc}} = -i\omega\epsilon_o\mu_o\mathbf{E}(\mathbf{r})_{\text{inc}}$, so that Eq. (2) yields

$$\sigma(\mathbf{r}) = 2\epsilon_o\hat{\mathbf{n}}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})_{\text{inc}}, \tag{58}$$

e.g., the local surface charge density is determined by the incident electric field at that point.

8. CONCLUSIONS

We have considered scattering of radiation by a surface or object of arbitrary shape, as schematically illustrated in Fig. 1. Inside the material there may be a charge density and current density, and on the boundary there may be a surface charge density and a surface current density. Integrals of Maxwell's Eqs. (3)–(6) are given by Eqs. (15) and (16) for the electric and magnetic fields, respectively. The solutions are expressed in terms of $\mathbf{\Pi_j}(\mathbf{r})$ and $\mathbf{\Pi_i}(\mathbf{r})$, which are determined by $\mathbf{j}(\mathbf{r})$ in the material and $\mathbf{i}(\mathbf{r})$ on the surface, through Eqs. (9) and (13), respectively. Equations (15) and (16) then express the solution as the sum of the incident field and the field generated by the current densities $\mathbf{j}(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})$. With Eqs. (12) and (14) we then easily verify that Eqs. (15) and (16) are indeed solutions of Maxwell's equations, both inside and outside the material.

Much less trivial is to show that the solutions of Eqs. (15) and (16) also satisfy the boundary conditions of Eqs.

(7) and (8) for any $\mathbf{j}(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})$. In Section 3 we considered the solution for $\mathbf{B}(\mathbf{r})$, given by Eq. (16). We let the field point approach a point on the surface, either from the inside or outside, as shown in Fig. 3. It turned out that the term $\nabla\times\mathbf{\Pi_i}(\mathbf{r})$, given by Eq. (22), is discontinuous across the surface. The integral in Eq. (22) is the same as in Eq. (29), and we see that the first term depends on from which side we approach the surface, and this difference gives the boundary conditions of Eq. (8) for $\mathbf{B}(\mathbf{r})$. The boundary condition for the electric field, Eq. (7), involves the surface charge density $\sigma(\mathbf{r})$, which does not appear explicitly in the solution of Eq. (15) for $\mathbf{E}(\mathbf{r})$. We found that both the terms $\nabla[\nabla\cdot\mathbf{\Pi_j}(\mathbf{r})]$ and $\nabla[\nabla\cdot\mathbf{\Pi_i}(\mathbf{r})]$ have a discontinuity across the surface, and when combined with the help of Eq. (2), we obtain the boundary condition of Eq. (7) for the electric field.

For a particular application, the solutions of Eqs.(15) and (16) have to be supplemented with a constitutive equation for the material under consideration. For a dielectric or metallic medium, the current density and the electric field are related by Eq. (50), which turns the solution of Eq. (15) into Eq. (51). The result is an integral equation for the electric field. When the material is a perfect conductor, the magnetic field inside is zero, and with Eq. (16) this leads to Eq. (55), which is an equation for the surface current density. When the surface of the perfect conductor is flat, the solution to Eq. (55) is given by Eq. (56), which expresses the surface current density in terms of the incident magnetic field at that same point. The surface charge density was similarly found to be given by Eq. (58).

APPENDIX A

Equation (2) expresses conservation of charge at the boundary, which involves the surface divergence of the current density. In this appendix we briefly summarize the definition of a surface divergence, and we present an identity that we need for the derivation in Section 4. More details can be found in Ref. 24.

Let the surface S be parametrized by $\mathbf{r}=\mathbf{r}(u^1,u^2)$, with u^1 and u^2 free parameters. For a given point (u^1,u^2) on the surface, the vectors

$$\mathbf{c}_k = \partial_k \mathbf{r}, \quad k = 1,2, \tag{A1}$$

span the tangent plane. Here, ∂_k is short for $\partial/\partial u^k$. The components of the fundamental tensor of the surface are $g_{ij}=\mathbf{c}_i\cdot\mathbf{c}_j$, and these can be seen as the elements of a symmetric 2×2 matrix. We then indicate by $g^{\ell k}$ the elements of the inverse matrix, so that $g^{\ell k}g_{ki}=\delta_i^\ell$ with a summation over k implied. The reciprocal basis vectors are defined as

$$\mathbf{c}^\ell = g^{\ell k}\mathbf{c}_k, \tag{A2}$$

and it then follows that

$$\mathbf{c}^i \cdot \mathbf{c}_j = \delta^i_j. \tag{A3}$$

The Christoffel symbols of the surface are defined by

$$\left\{ \begin{matrix} & k & \\ j & & i \end{matrix} \right\} = \mathbf{c}^k \cdot \partial_j \mathbf{c}_i, \tag{A4}$$

which are functions of u^1 and u^2.

A vector $\mathbf{T}$ in the tangent plane can be written as

$$\mathbf{T} = T^k \mathbf{c}_k, \tag{A5}$$

so that with Eq. (A3)

$$T^k = \mathbf{c}^k \cdot \mathbf{T}, \tag{A6}$$

which are the contravariant components of $\mathbf{T}$. When a vector $\mathbf{T}$ is associated with each point on the surface, it is a vector field, and its surface divergence is defined as

$$\nabla_S \cdot \mathbf{T} = \partial_k T^k + \left\{ \begin{matrix} & j & \\ k & & j \end{matrix} \right\} T^k. \tag{A7}$$

Let $\phi(\mathbf{r})$ be a scalar field in space. With the surface embedded in space, this makes it also a scalar field on S. Then $\phi\mathbf{T}$ is a vector field on the surface, with contravariant components $(\phi\mathbf{T})^k=\phi\mathbf{T}^k$. The surface divergence of $\phi\mathbf{T}$ is then, with Eq. (A7),

$$\nabla_S \cdot (\phi\mathbf{T}) = \phi\nabla_S \cdot \mathbf{T} + T^k\partial_k\phi. \tag{A8}$$

The summation on the right-hand side runs over $k=1,2$. On the other hand, $\nabla\phi$ is a vector field with three components, and $\nabla\phi$ will in general not be in the tangent plane. With the chain rule we have

$$\partial_k\phi = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial u^k} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial u^k} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial u^k}, \tag{A9}$$

and when we write out Eq. (A1) in Cartesian coordinates we have

$$\mathbf{c}_k = \mathbf{e}_x\frac{\partial x}{\partial u^k} + \mathbf{e}_y\frac{\partial y}{\partial u^k} + \mathbf{e}_z\frac{\partial z}{\partial u^k}. \tag{A10}$$

Therefore,

$$\partial_k\phi = \mathbf{c}_k \cdot \nabla\phi, \tag{A11}$$

since $\nabla\phi=\mathbf{e}_x(\partial\phi/\partial x)+\mathbf{e}_y(\partial\phi/\partial y)+\mathbf{e}_z(\partial\phi/\partial z)$. With Eq. (A5) we have $\mathbf{T}\cdot\nabla\phi=T^k(\mathbf{c}_k\cdot\nabla\phi)$, with $k=1,2$, and so we have $\mathbf{T}\cdot\nabla\phi=T^k\partial_k\phi$. Equation (A8) then becomes

$$\nabla_S \cdot (\phi\mathbf{T}) = \phi\nabla_S \cdot \mathbf{T} + \mathbf{T}\cdot\nabla\phi. \tag{A12}$$

We have used this identity in the step from Eq. (36) to Eq. (37), with $\mathbf{T}$ the surface current density and ϕ the Green's function.

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