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Conservation of charge at an interface

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Abstract

In a continuous medium, the charge density and the current density are related by the continuity equation, expressing conservation of charge. When an interface separates two continuous media, we can also have a surface charge density σ and a surface current density $\mathbf{i}$ on the interface surface. We consider an interface of arbitrary shape, and derive the continuity equation relating σ and $\mathbf{i}$ from the principle of conservation of charge. The result can be expressed in terms of the surface divergence of $\mathbf{i}$ on the surface, familiar from tensor analysis. In an independent approach, we derive the same equation from Maxwell's equations at the boundary.
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1. Introduction

When electromagnetic radiation scatters off a surface, a charge density $\rho(\mathbf{r},t)$ and current density $\mathbf{j}(\mathbf{r},t)$ are induced in the material and a surface charge density $\sigma(\mathbf{r},t)$ and surface current density $\mathbf{i}(\mathbf{r},t)$ may appear on the surface of the material. We shall consider the boundary, or interface, between two continuous media, and we shall allow the shape of the boundary to be arbitrary (that is, not flat). In a continuous medium, the electric field $\mathbf{E}(\mathbf{r},t)$ and magnetic field $\mathbf{B}(\mathbf{r},t)$ are related to the charge and current densities $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ as given by Maxwell's equations:

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\varepsilon_o}, \tag{1}$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}, \tag{2}$$

$$\nabla\cdot\mathbf{B}=0, \tag{3}$$

$$\nabla\times\mathbf{B}=\mu_o\mathbf{j}+\varepsilon_o\mu_o\frac{\partial\mathbf{E}}{\partial t}. \tag{4}$$

By taking the divergence of (4) and using (1) we obtain

$$\nabla\cdot\mathbf{j}=-\frac{\partial\rho}{\partial t}, \tag{5}$$

the continuity equation, expressing the conservation of charge.

Across a boundary between two continuous media, the fields $\mathbf{E}$, $\mathbf{B}$, ρ and $\mathbf{j}$ can be discontinuous, leaving the operations div and curl in Eqs. (1)–(5) undetermined. To find Maxwell's equations for a field point $\mathbf{r}$ on the boundary between two continuous media, we integrate (1) and (3) over a volume V containing $\mathbf{r}$, as shown in Fig. 1, and include the possibility of a surface charge density. We then use the divergence theorem and then let the volume shrink to a Gaussian pillbox. For Eqs. (2) and (4) we similarly integrate over a Stokessian loop around $\mathbf{r}$. This familiar procedure [1] leads to four equations, replacing Eqs. (1)–(4). The four equations can be combined into two as

$$\mathbf{E}_2-\mathbf{E}_1=\frac{\sigma}{\varepsilon_o}\hat{\mathbf{n}}, \tag{6}$$

$$\mathbf{B}_2-\mathbf{B}_1=\mu_o\mathbf{i}\times\hat{\mathbf{n}}, \tag{7}$$

with $\hat{\mathbf{n}}$ the unit normal vector on the surface, directed from medium 1 to medium 2. Here, $\mathbf{E}_1$ ($\mathbf{B}_1$) and $\mathbf{E}_2$ ($\mathbf{B}_2$) are the

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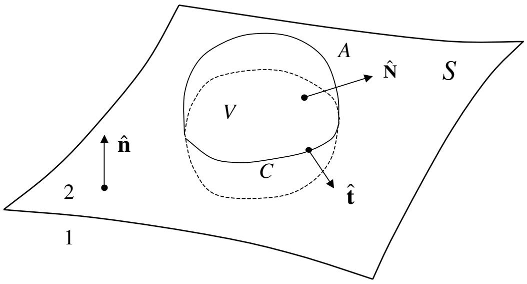

Fig. 1. The surface S separates medium 1 from medium 2, and the normal vector $\hat{\mathbf{n}}$ on S points from 1 to 2. Volume V with boundary surface A is partially in medium 1 and partially in medium 2, and the intersection between the surfaces A and S is the curve C. The normal vector on A is indicated by $\hat{\mathbf{N}}$, and it points to the outside of the volume. The unit vector $\hat{\mathbf{t}}$ is perpendicular to C, directed towards the outside, and lies in the local tangent plane to S.

values of the electric (magnetic) field just off the surface in medium 1 and 2, respectively. The right-hand side of Eq. (6) is proportional to $\hat{\mathbf{n}}$, and has no component parallel to the surface. Therefore, Eq. (6) implies that the parallel component of $\mathbf{E}$ is continuous across the surface, which is usually referred to as a boundary condition. Eq. (6) also implies that the electric field jumps with $\sigma\hat{\mathbf{n}}/\varepsilon_o$ across the boundary. Eq. (7) can be read in a similar way.

The continuity equation (5) relates the charge and current densities ρ and $\mathbf{j}$ in a continuous medium, and this equation follows directly from Maxwell's equations (1) and (4). At the interface, Maxwell's equations take the form of Eqs. (6) and (7), now involving the surface charge and current densities σ and $\mathbf{i}$. One would then expect that an equation similar to (5) could be derived, expressing conservation of charge at the boundary, and thereby relating σ and $\mathbf{i}$. It appears that this step is not made in the literature. In this communication we shall derive the equivalent of (5) for a point on the boundary.

2. Conservation of charge

A surface S separates two media, labeled 1 and 2. We consider a volume V with boundary surface A, which is partially in medium 1 and partially in medium 2, as illustrated in Fig. 1. The intersection between A and S is a curve C on S. The unit normal vector on A will be indicated by $\hat{\mathbf{N}}$, and the unit normal on the interface surface S by $\hat{\mathbf{n}}$. Vector $\hat{\mathbf{t}}$ is a unit vector, perpendicular to C, directed outward, and in the tangent plane of S. Conservation of charge then requires that the charge flowing out of V through A per unit of time is equal to the loss rate of the charge inside V. Expressed in terms of $\mathbf{j}$, $\mathbf{i}$, ρ and σ this becomes

$$\oint_A \mathbf{j}\cdot\hat{\mathbf{N}}\,\mathrm{d}A + \oint_C \mathbf{i}\cdot\hat{\mathbf{t}}\,\mathrm{d}s = -\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathrm{d}V - \frac{\mathrm{d}}{\mathrm{d}t}\int_S \sigma\,\mathrm{d}S. \tag{8}$$

Here, ds is the infinitesimal arc length of C, and the second integral on the right-hand side runs over the part of the interface which is enclosed by C.

We now take the volume V as in Fig. 2. The top and the bottom follow the curving of S, with the top just in medium 1 and the bottom just in medium 2. The height Δh, separating the top and the bottom, will be considered very small, and approaching zero. Therefore, the volume V goes to zero for $\Delta h \to 0$, and so $\int \rho\,\mathrm{d}V \to 0$. For the first integral on the left-hand side of Eq. (8), only the top and the bottom contribute in the limit $\Delta h \to 0$, and so Eq. (8) reduces to

$$\int_S (\mathbf{j}_2 - \mathbf{j}_1)\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \oint_C \mathbf{i}\cdot\hat{\mathbf{t}}\,\mathrm{d}s = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \sigma\,\mathrm{d}S. \tag{9}$$

Here $\mathbf{j}_1$ and $\mathbf{j}_2$ are the current densities in medium 1 and 2, respectively, and evaluated just off the surface. Eq. (9) only involves fields along the surface, and is the integral form of conservation of charge on an interface. It is interesting to see that, besides $\mathbf{i}$ and σ, also the volume current density $\mathbf{j}$ appears.

In order to arrive at a differential form similar to Eq. (5), we let S shrink to ΔS, and consider the limit $\Delta\, S \to 0$. For the first and the last integrals in Eq. (9) we simply set $\mathrm{d}S \to \Delta S$, and leave out the integral signs. The integral containing $\mathbf{i}$ represents a complication. We now define the quantity $\nabla_S \cdot \mathbf{i}$ as follows:

$$\nabla_S \cdot \mathbf{i} = \frac{1}{\Delta S}\oint_C \mathbf{i}\cdot\hat{\mathbf{t}}\,\mathrm{d}s, \quad \Delta S \to 0. \tag{10}$$

Here the contour C is the boundary of ΔS. The continuity equation then becomes

$$\nabla_S \cdot \mathbf{i} + (\mathbf{j}_2 - \mathbf{j}_1)\cdot\hat{\mathbf{n}} = -\frac{\partial \sigma}{\partial t}. \tag{11}$$

Apart from the appearance of $\mathbf{j}$, this equation is identical in form to Eq. (5). The quantity $\nabla_S \cdot \mathbf{i}$, as defined by Eq. (10), has yet to be determined. It is written as a divergence, but since $\mathbf{i}$ is a vector which is only defined in the tangent plane of S, this divergence cannot be the same as the usual divergence of a vector field in space (as in Eqs. (1), (3) and (5)). The notation is inspired by the coordinate independent definition of the regular divergence [2], which has a similar appearance. Furthermore, for a flat surface the divergence theorem in two dimensions holds [3], from which (10) fol-

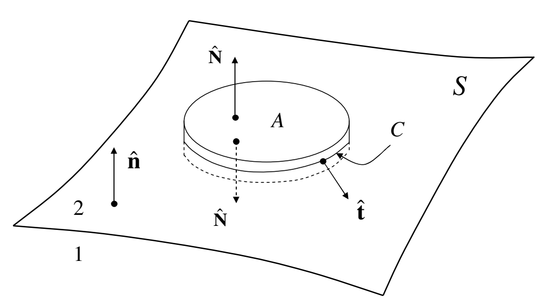

Fig. 2. The volume V of Fig. 1 is now taken as illustrated in the figure. The top and the bottom of A are in medium 2 and 1, respectively, they closely follow the shape of the surface S, and they are separated by a negligible distance. For the top surface we then have $\hat{\mathbf{N}} = \hat{\mathbf{n}}$ and for the bottom we have $\hat{\mathbf{N}} = -\hat{\mathbf{n}}$. By setting the current flowing out of the volume equal to the loss rate of the charge inside the volume, we arrive at Eq. (9), which is the integral form of conservation of charge on the surface S.

lows as a coordinate independent definition of the regular $\nabla \cdot \mathbf{i}$. Therefore, for a flat surface the $\nabla_S \cdot \mathbf{i}$ defined by Eq. (10) is the usual two-dimensional divergence, and when we take the surface as the xy-plane we have $\nabla_S \cdot \mathbf{i} = \partial i_x/\partial x + \partial i_y/\partial y$.

3. The tangent plane

A point in space can be represented by its Cartesian coordinates (x, y, z), or equivalently by its position vector $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$. The collection of points that form the surface S can then be parametrized as $x = x(u^1, u^2)$, $y = y(u^1, u^2)$ and $z = z(u^1, u^2)$, with u^1 and u^2 free parameters, or equivalently as $\mathbf{r} = \mathbf{r}(u^1, u^2)$. A parameter curve on S is defined as the curve that results if we keep one parameter fixed in $\mathbf{r} = \mathbf{r}(u^1, u^2)$ while varying the other. The partial derivatives of $\mathbf{r}(u^1, u^2)$ with respect to u^1 and u^2 are tangent vectors of the parameter curves, and we shall indicate these by $\mathbf{c}_1$ and $\mathbf{c}_2$, respectively. With the notation ∂_i as short for $\partial/\partial u^i$, these vectors are

$$\mathbf{c}_i = \partial_i \mathbf{r}(u^1, u^2). \tag{12}$$

For a given point on S, these vectors span the tangent plane, and as usual we consider the given point on S as the origin of coordinates for the tangent plane. Any vector $\mathbf{T}$ in the tangent plane is then a linear combination of $\mathbf{c}_1$ and $\mathbf{c}_2$:

$$\mathbf{T} = T^i \mathbf{c}_i. \tag{13}$$

We shall adopt the Einstein summation convention in which a summation over an index is implied if it appears as both a superscript and a subscript in a formula. The two numbers T^i are called the contravariant components of vector $\mathbf{T}$. The surface current density $\mathbf{i}$ under consideration here is such a vector, since at any point on the surface this $\mathbf{i}$ is in the local tangent plane. Fig. 3 illustrates the situation.

Vector fields in space, like $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{j}$, will also have a value near a point on S. Their parallel components are vectors in the tangent plane, and their perpendicular components are directed along or opposite to the normal vector

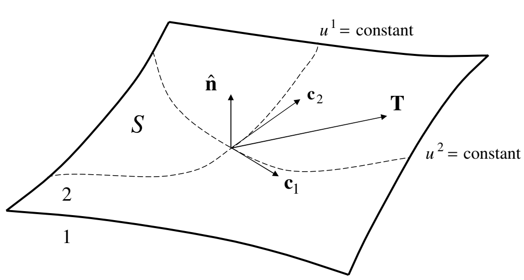

Fig. 3. The surface S is parametrized with the free parameters u^1 and u^2. For a given point on the surface, two parameter curves intersect, and the vectors $\mathbf{c}_1$ and $\mathbf{c}_2$ are the tangent vectors to these parameter curves at that point. Vectors $\mathbf{c}_1$ and $\mathbf{c}_2$ span the local tangent plane, so that an arbitrary vector $\mathbf{T}$ in the tangent plane is a linear combination of $\mathbf{c}_1$ and $\mathbf{c}_2$.

$\hat{\mathbf{n}}$ at this point. Since $\hat{\mathbf{n}}$ is perpendicular to both $\mathbf{c}_1$ and $\mathbf{c}_2$, it is given by

$$\hat{\mathbf{n}} = \frac{\pm 1}{|\mathbf{c}_1 \times \mathbf{c}_2|} \mathbf{c}_1 \times \mathbf{c}_2. \tag{14}$$

The ± 1 indicates the two possible orientations of $\hat{\mathbf{n}}$. We shall leave this orientation unspecified and carry the ± 1 along, keeping in mind that in an actual problem one usually makes a choice.

4. Fundamental tensor

The fundamental tensor of S is defined as the usual dot product in space. Its covariant components g_{ij} with respect to the basis $\mathbf{c}_1$, $\mathbf{c}_2$ are by definition

$$g_{ij} = \mathbf{c}_i \cdot \mathbf{c}_j, \tag{15}$$

and these four numbers are functions of the parameters u^1 and u^2. They can also be seen as the matrix elements of a symmetric 2×2 matrix G. The contravariant components of the fundamental tensor are indicated by $g^{k\ell}$, and these four numbers are by definition the solution of the set of equations

$$g^{k\ell} g_{\ell i} = \delta^k_i \tag{16}$$

with δ^k_i the Kronecker delta. Or, the $g^{k\ell}$ are the matrix elements of the inverse matrix of G.

We then define vectors $\mathbf{c}^1$ and $\mathbf{c}^2$ as

$$\mathbf{c}^i = g^{ij} \mathbf{c}_j. \tag{17}$$

They are linear combinations of $\mathbf{c}_1$ and $\mathbf{c}_2$, and they form what is called the reciprocal basis. From Eqs. (16) and (17) it follows that

$$\mathbf{c}^i \cdot \mathbf{c}_k = \delta^i_k, \tag{18}$$

so the reciprocal basis vector $\mathbf{c}^1$ ($\mathbf{c}^2$) is perpendicular to the regular basis vector $\mathbf{c}_2$ ($\mathbf{c}_1$). From Eqs. (13) and (18) we see that the contravariant components of $\mathbf{T}$ are given by

$$T^i = \mathbf{c}^i \cdot \mathbf{T}, \tag{19}$$

e.g., they are the projections of $\mathbf{T}$ onto the reciprocal basis. A vector $\mathbf{T}$ can be represented with respect to the reciprocal basis as

$$\mathbf{T} = T_i \mathbf{c}^i, \tag{20}$$

where the T_i are the covariant components of $\mathbf{T}$, and with Eq. (18) we see that

$$T_i = \mathbf{c}_i \cdot \mathbf{T}. \tag{21}$$

The covariant and contravariant components of $\mathbf{T}$ are related as $T_j = g_{ji} T^i$, and the contravariant components of the fundamental tensor are also given by $g^{ij} = \mathbf{c}^i \cdot \mathbf{c}^j$.

The determinant of the matrix G, representing the fundamental tensor, is $\det G = g_{11} g_{22} - g_{12} g_{21}$, and one verifies by inspection that this is equal to

$$\det G = |\mathbf{c}_1 \times \mathbf{c}_2|^2. \tag{22}$$

5. Differentiation in the tangent plane

When we write a vector in the tangent plane as $\mathbf{T} = T^i\mathbf{c}_i$, then both the components T^i and the basis vectors $\mathbf{c}_i$ are functions of the parameters u^1 and u^2. The parameter dependence of T^i can, of course, be anything, but the parameter dependence of the basis vectors is determined by the shape of S, e.g., by the function $\mathbf{r}(u^1, u^2)$. If one needs to differentiate $\mathbf{T}$ with respect to u^1 or u^2, then one needs the derivatives of $\mathbf{c}_i$. The derivative of $\mathbf{c}_i$ is not necessarily in the tangent plane, and as a matter of notation we write

$$\partial_j\mathbf{c}_i = \left\{ {k \atop j \quad i} \right\}\mathbf{c}_k + h_{ji}\hat{\mathbf{n}}. \tag{23}$$

The components of this vector $\partial_j\mathbf{c}_i$ in the tangent plane, the $\left\{ {k \atop j \quad i} \right\}$, are called Christoffel symbols (of the second kind), and the perpendicular component h_{ji} of the vector $\partial_j\mathbf{c}_i$ is a covariant component of the second fundamental tensor. With the function $\mathbf{r}(u^1, u^2)$ given, this is $\partial_j\mathbf{c}_i = \partial_j\,\partial_i\mathbf{r}(u^1, u^2)$, so that the Christoffel symbols and the h_{ji}'s are determined by knowledge of the shape of the surface. We see that the Christoffel symbols are symmetric in their lower indices, because $\partial_j\mathbf{c}_i = \partial_i\mathbf{c}_j$, and that $h_{ij} = h_{ji}$. From Eqs. (23) and (18) we have explicitly

$$\left\{ {k \atop j \quad i} \right\} = \mathbf{c}^k \cdot \partial_j\mathbf{c}_i. \tag{24}$$

6. Evaluation of $\nabla_S \cdot \mathbf{T}$

Let us now consider a vector field $\mathbf{T}$, defined on the surface S, and $\mathbf{T}$ is in the tangent plane at every point on the surface. The quantity $\nabla_S \cdot \mathbf{T}$ is then defined as, Eq. (10),

$$\nabla_S \cdot \mathbf{T} = \frac{1}{\Delta S}\oint_C \mathbf{T} \cdot \hat{\mathbf{t}}\,\mathrm{d}s, \quad \Delta S \to 0. \tag{25}$$

For a point on the surface with parameters u^1, u^2 we take ΔS as the part of the surface where u^1 runs from u^1 to $u^1 + \Delta u^1$ and u^2 runs from u^2 to $u^2 + \Delta u^2$, as shown in Fig. 4. The vector representing the side AB is then approximately $\mathbf{r}_B - \mathbf{r}_A$, and with $\mathbf{r}_B \approx \mathbf{r}_A + (\partial_1\mathbf{r})_A\Delta u^1$ we find from Eq. (12) $\mathbf{r}_B - \mathbf{r}_A \approx \mathbf{c}_1\Delta\, u^1$. The side AD in the figure is similarly represented by $\mathbf{c}_2\Delta u^2$. Since the area ΔS is approximately a parallelogram, the surface area is

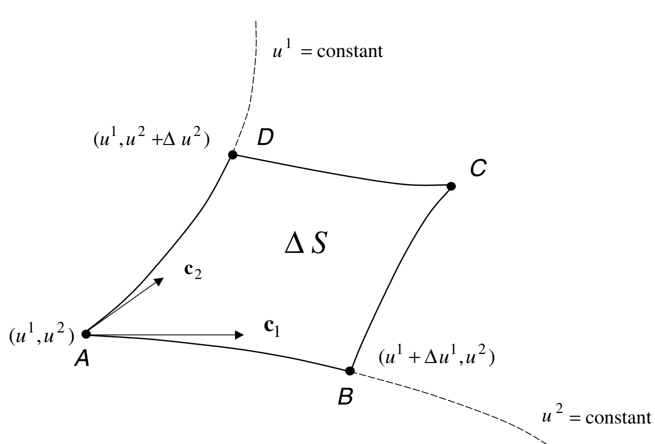

Fig. 4. In order to evaluate $\nabla_S \cdot \mathbf{T}$, as defined by Eq. (25), we consider the small surface area ΔS shown in the figure.

$$\Delta S \approx |\mathbf{c}_1 \times \mathbf{c}_2|\Delta u^1\Delta u^2, \tag{26}$$

and with (22) this is

$$\Delta S \approx \sqrt{\det G}\Delta u^1\Delta u^2. \tag{27}$$

The vector $\hat{\mathbf{t}}$ in Eq. (25) is a unit vector in the tangent plane, perpendicular to C and directed to the outside of the loop. For the side AD this vector must be perpendicular to $\mathbf{c}_2$. From Eq. (18) we see that $\mathbf{c}^1 \cdot \mathbf{c}_2 = 0$, so that $\mathbf{c}^1$ is perpendicular to $\mathbf{c}_2$. Therefore, $\hat{\mathbf{t}}$ must be proportional to $\mathbf{c}^1$. From Eq. (17) we derive the explicit form

$$\mathbf{c}^1 = \frac{1}{\det G}(g_{22}\mathbf{c}_1 - g_{12}\mathbf{c}_2). \tag{28}$$

Since $\det G > 0$ and $g_{22} = \mathbf{c}_2 \cdot \mathbf{c}_2 > 0$ we observe that the $\mathbf{c}_1$ component of $\mathbf{c}^1$ is positive, and therefore $\hat{\mathbf{t}}$ must be in the opposite direction of $\mathbf{c}^1$. From Eqs. (18) and (28) we have $\mathbf{c}^1 \cdot \mathbf{c}^1 = g_{22}/\det G$, and therefore

$$\hat{\mathbf{t}} \approx -\mathbf{c}^1\sqrt{\frac{\det G}{g_{22}}} \tag{29}$$

for the side AD. We write $\approx$ here, instead of an equal sign. Eq. (29) is exact for point A, but an approximation for other points on the side AD. For the length of the side AD we have $\Delta s \approx |\mathbf{c}_2|\Delta u^2$, which is $\Delta s \approx \sqrt{g_{22}}\Delta u^2$, since $g_{22} = \mathbf{c}_2 \cdot \mathbf{c}_2$. Therefore, we have for the side AD

$$\hat{\mathbf{t}}\Delta s \approx -\mathbf{c}^1\sqrt{\det G}\,\Delta u^2. \tag{30}$$

If we now take the dot product with $\mathbf{T}$, we find with Eq. (19)

$$\mathbf{T} \cdot \hat{\mathbf{t}}\Delta s \approx -\left(T^1\sqrt{\det G}\right)_A\Delta u^2. \tag{31}$$

The T^1 and G depend on u^1 and u^2, and here we have indicated explicitly that they have to be evaluated at point A.

For the contribution of the side BC to the integral, everything is the same, except that $\hat{\mathbf{t}}$ must be taken in the opposite direction, as compared to the $\hat{\mathbf{t}}$ in Eq. (29). For the side BC we have to evaluate T^1 and G at point B, for which

$$\left(T^1\sqrt{\det G}\right)_B \approx \left(T^1\sqrt{\det G}\right)_A + \Delta u^1\partial_1\left(T^1\sqrt{\det G}\right)_A. \tag{32}$$

Therefore, along BC

$$\mathbf{T} \cdot \hat{\mathbf{t}}\Delta s \approx \left[\left(T^1\sqrt{\det G}\right)_A + \Delta u^1\partial_1\left(T^1\sqrt{\det G}\right)_A\right]\Delta u^2. \tag{33}$$

When we add Eqs. (31) and (33) we notice that the first term on the right-hand side of Eq. (33) cancels against the right-hand side of Eq. (31). In a similar way we obtain the contributions to the integral from sides AB and DC. For the total integral around the loop we then find

$$\oint_C \mathbf{T}\cdot\hat{\mathbf{t}}\,\mathrm{d}s \approx \partial_i\left(T^i\sqrt{\det G}\right)\Delta u^1\Delta u^2. \tag{34}$$

Then we divide by $\Delta S \approx \sqrt{\det G}\Delta u^1\Delta u^2$, which then yields

$$\nabla_S\cdot\mathbf{T} = \frac{1}{\sqrt{\det G}}\partial_i\left(T^i\sqrt{\det G}\right), \tag{35}$$

as an exact result in the limit $\Delta S \to 0$. We note that the evaluation of $\nabla_S\cdot\mathbf{T}$ involves the contravariant components of $\mathbf{T}$, which are $T^i = \mathbf{c}^i\cdot\mathbf{T}$.

Apparently, the quantity $\nabla_S\cdot\mathbf{T}$, as defined by Eq. (25), is a meaningful quantity, and Eq. (35) explicitly expresses how $\nabla_S\cdot\mathbf{T}$ can be evaluated for an arbitrary coordinate system of the surface. The same holds of course for $\nabla_S\cdot\mathbf{i}$ in Eq. (10), and therefore Eq. (11) is the proper differential form of conservation of charge at an interface. If the surface is flat, taken as the xy-plane, then the tangent plane is the xy-plane itself, and we have $u^1 = x$, $u^2 = y$. Then $\mathbf{c}_1 = \partial_x\mathbf{r} = \mathbf{e}_x$ and $\mathbf{c}_2 = \partial_y\mathbf{r} = \mathbf{e}_y$, and G is the unit matrix with $\det G = 1$. With $g_{ji} = \delta_{ji}$ and $T_j = g_{ji}T^i$, the distinction between contravariant and covariant components of a vector disappears, and therefore Eq. (35) reduces to $\nabla_S\cdot\mathbf{T} = \partial_i T^i$, which is the regular divergence in two dimensions.

The result in Eq. (35) can also be expressed as

$$\nabla_S\cdot\mathbf{T} = \partial_i T^i + \frac{T^i}{2\det G}\partial_i \det G. \tag{36}$$

With the determinant of G given by Eq. (22), its derivatives with respect to the surface parameters u^1 and u^2 can be evaluated by means of Eq. (23). The result is

$$\partial_i \det G = 2\det G\left\{\begin{matrix} & j & \\ i & & j\end{matrix}\right\}. \tag{37}$$

The alternative expression for $\nabla_S\cdot\mathbf{T}$ then becomes

$$\nabla_S\cdot\mathbf{T} = \partial_i T^i + \left\{\begin{matrix} & j & \\ i & & j\end{matrix}\right\}T^i. \tag{38}$$

In order to use this expression, one needs to know the Christoffel symbols of the surface, given by Eq. (24), rather than only the determinant of G.

Our two-parameter surface S, embedded in three dimensional space, is a simple example of a Riemannian manifold [4]. In the general theory of tensor analysis on manifolds with a metric (a Riemann space) one defines the covariant derivative of a contravariant component of a vector field defined in the tangent space of the manifold by (p. 71 of Ref. [4], or Ref. [5])

$$T^i_{;k} = \partial_k T^i + \left\{\begin{matrix} & i & \\ k & & j\end{matrix}\right\}T^j. \tag{39}$$

In terms of this covariant derivative, we can write $\nabla_S\cdot\mathbf{T}$ as

$$\nabla_S\cdot\mathbf{T} = T^i_{;i}, \tag{40}$$

which is the tensor contraction of $T^i_{;k}$. In the literature, this contracted covariant derivative is usually simply called the divergence of $\mathbf{T}$. Since in our context we also have the usual divergence in three-dimensional space, we shall call $\nabla_S\cdot\mathbf{T}$ the surface divergence of $\mathbf{T}$ on S.

7. Derivation from Maxwell's equations

In a continuous medium, the continuity equation (5) follows immediately from Maxwell's equations in the medium. At the boundary, Maxwell's equations are given by Eqs. (6) and (7), and one would expect that it should also be possible to derive the continuity equation at the boundary, Eq. (11), directly from Maxwell's equations at the boundary. To this end we first take the cross product with $\hat{\mathbf{n}}$ on both sides of Eq. (7), which yields

$$\mathbf{i} = \frac{1}{\mu_o}\hat{\mathbf{n}}\times(\mathbf{B}_2 - \mathbf{B}_1). \tag{41}$$

From the result of the previous section we know that we should consider the surface divergence of $\mathbf{i}$ on S. On the right-hand side, the $\mathbf{B}_1$ and $\mathbf{B}_2$ are the values of the magnetic field $\mathbf{B}$ just off the surface in medium 1 and 2, respectively, and these vectors do not lie in the local tangent plane. The right-hand side of Eq. (41), however, is a vector in the tangent plane, due to the cross product with $\hat{\mathbf{n}}$, and we can consider its surface divergence. In Appendices A and B, C we derive the following theorem. Let $\mathbf{F}$ be a vector field in space. Then $\mathbf{F}$ also has a value on and near the surface S. The theorem then states that the surface divergence of $\hat{\mathbf{n}}\times\mathbf{F}$, which is a vector in the tangent plane, can be expressed as

$$\nabla_S\cdot(\hat{\mathbf{n}}\times\mathbf{F}) = -\hat{\mathbf{n}}\cdot(\nabla\times\mathbf{F}), \tag{42}$$

where $\nabla\times\mathbf{F}$ on the right-hand side is the usual curl of $\mathbf{F}$ in three dimensions. With this theorem we find from Eq. (41):

$$\nabla_S\cdot\mathbf{i} = -\frac{1}{\mu_o}\hat{\mathbf{n}}\cdot(\nabla\times\mathbf{B}_2 - \nabla\times\mathbf{B}_1). \tag{43}$$

Since $\mathbf{B}_1$ and $\mathbf{B}_2$ are the values of the magnetic field just off the surface, they satisfy Maxwell's equations (1)–(4). With Eq. (4), we then obtain

$$\nabla_S\cdot\mathbf{i} = -\hat{\mathbf{n}}\cdot(\mathbf{j}_2 - \mathbf{j}_1) - \varepsilon_o\frac{\partial}{\partial t}\hat{\mathbf{n}}\cdot(\mathbf{E}_2 - \mathbf{E}_1). \tag{44}$$

Then we substitute the right-hand side of Eq. (6) for $\mathbf{E}_2 - \mathbf{E}_1$, which yields

$$\nabla_S\cdot\mathbf{i} = -\hat{\mathbf{n}}\cdot(\mathbf{j}_2 - \mathbf{j}_1) - \frac{\partial\sigma}{\partial t}, \tag{45}$$

and this is the continuity equation at the boundary.

8. Conclusions

In a continuous medium we have a charge density $\rho(\mathbf{r},t)$ and current density $\mathbf{j}(\mathbf{r},t)$, which are related by the continuity equation (5). This equation can be derived from conservation of charge, by setting the current flowing out of a volume equal to the loss rate of the charge inside the volume, or it can be derived from Maxwell's equations (1) and (4) for a continuous medium. We have considered an interface between two media. Then, at the interface, there may appear a surface charge density $\sigma(\mathbf{r},t)$ and a surface current density $\mathbf{i}(\mathbf{r},t)$. Maxwell's equations for a point on the interface take the form of Eqs. (6) and (7), involving $\sigma(\mathbf{r},t)$ and $\mathbf{i}(\mathbf{r},t)$. Also the continuity equation (5) will take a different form for a point on the interface.

By considering conservation of charge, we have shown that the continuity equation at the interface is given by Eq. (11), involving the quantity $\nabla_S \cdot \mathbf{i}$, defined by Eq. (10). In Section 6 we have evaluated this quantity for an arbitrary vector field $\mathbf{T}$ in the tangent plane of the boundary surface between the two media. The resulting expression for $\nabla_S \cdot \mathbf{T}$ is given by Eq. (35) in terms of the determinant of the matrix G, containing the covariant components of the fundamental tensor of the surface, and the contravariant components T^i of the vector field $\mathbf{T}$. Alternatively, the surface divergence $\nabla_S \cdot \mathbf{T}$ could be expressed in the form given by Eq. (38), now involving the Christoffel symbols of the surface.

Just as Eq. (5) could be derived from Maxwell's equations for a continuous medium, we have derived the continuity equation (11) for a point on the interface from Maxwell's equations at the interface, Eqs. (6) and (7), in Section 7. For this derivation we needed the theorem given by Eq. (42), where $\mathbf{F}$ is a vector field near the boundary, evaluated at the boundary. The theorem relates the surface divergence of the vector field $\hat{\mathbf{n}} \times \mathbf{F}$, which is a vector field in the tangent plane, to the curl of $\mathbf{F}$ just off the boundary.

Appendix A

The derivation of the continuity equation at the boundary from Maxwell's equations in Section 7 relies on the theorem given by Eq. (42). In this appendix we give the first step of its proof. We consider a vector field $\mathbf{F}$ in space. For a field point on the surface S, we can write $\mathbf{F}$ as

$$\mathbf{F} = F_j \mathbf{c}^j + F_\perp \hat{\mathbf{n}}, \tag{A.1}$$

where the first term on the right-hand side is the component of $\mathbf{F}$ in the tangent plane. Note that in the tangent plane we use the covariant components F_j of $\mathbf{F}$. Then

$$\hat{\mathbf{n}} \times \mathbf{F} = F_j \hat{\mathbf{n}} \times \mathbf{c}^j. \tag{A.2}$$

We now want to consider the surface divergence of the right-hand side of this equation, and we shall use the form given by Eq. (38). This expression involves the i^{th} contravariant component, which is, according to Eq. (19)

$$(\hat{\mathbf{n}} \times \mathbf{F})^i = F_j \mathbf{c}^i \cdot (\hat{\mathbf{n}} \times \mathbf{c}^j). \tag{A.3}$$

The reciprocal basis vectors are defined by Eq. (17), and the normal vector is given by Eq. (14). This gives $\mathbf{c}^1$ as given by Eq. (28), and a similar expression for $\mathbf{c}^2$ can be derived. With some manipulations we can then obtain the expressions

$$\mathbf{c}^1 = \frac{\pm 1}{\sqrt{\det G}} \mathbf{c}_2 \times \hat{\mathbf{n}}, \tag{A.4}$$

$$\mathbf{c}^2 = \frac{\mp 1}{\sqrt{\det G}} \mathbf{c}_1 \times \hat{\mathbf{n}}. \tag{A.5}$$

Taking the cross product with $\hat{\mathbf{n}}$ then yields

$$\hat{\mathbf{n}} \times \mathbf{c}^1 = \frac{\pm 1}{\sqrt{\det G}} \mathbf{c}_2, \tag{A.6}$$

$$\hat{\mathbf{n}} \times \mathbf{c}^2 = \frac{\mp 1}{\sqrt{\det G}} \mathbf{c}_1. \tag{A.7}$$

With Eq. (18) we then find

$$\mathbf{c}^2 \cdot (\hat{\mathbf{n}} \times \mathbf{c}^1) = \frac{\pm 1}{\sqrt{\det G}}, \tag{A.8}$$

$$\mathbf{c}^1 \cdot (\hat{\mathbf{n}} \times \mathbf{c}^2) = \frac{\mp 1}{\sqrt{\det G}}, \tag{A.9}$$

and $\mathbf{c}^1 \cdot (\hat{\mathbf{n}} \times \mathbf{c}^1) = \mathbf{c}^2 \cdot (\hat{\mathbf{n}} \times \mathbf{c}^2) = 0$. With Eq. (A.3) this gives

$$(\hat{\mathbf{n}} \times \mathbf{F})^1 = \frac{\mp 1}{\sqrt{\det G}} F_2, \tag{A.10}$$

$$(\hat{\mathbf{n}} \times \mathbf{F})^2 = \frac{\pm 1}{\sqrt{\det G}} F_1, \tag{A.11}$$

and with this

$$\partial_i (\hat{\mathbf{n}} \times \mathbf{F})^i = \pm \partial_2 \frac{F_1}{\sqrt{\det G}} \mp \partial_1 \frac{F_2}{\sqrt{\det G}}. \tag{A.12}$$

From Eq. (37) we find

$$\partial_k \frac{1}{\sqrt{\det G}} = -\frac{1}{\sqrt{\det G}} \left\{ \begin{matrix} & j & \\ k & & j \end{matrix} \right\}, \tag{A.13}$$

with which Eq. (A.12) becomes

$$\partial_i (\hat{\mathbf{n}} \times \mathbf{F})^i = \frac{\pm 1}{\sqrt{\det G}} \left(\partial_2 F_1 - \partial_1 F_2 - F_1 \left\{ \begin{matrix} & j & \\ 2 & & j \end{matrix} \right\} + F_2 \left\{ \begin{matrix} & j & \\ 1 & & j \end{matrix} \right\} \right). \tag{A.14}$$

We then substitute this as the first term on the right-hand side of Eq. (38), and for the second term on the right-hand side we use Eqs. (A.10) and (A.11). It then appears that all terms with Christoffel symbols cancel, and we find

$$\nabla_S \cdot (\hat{\mathbf{n}} \times \mathbf{F}) = \frac{\pm 1}{\sqrt{\det G}} (\partial_2 F_1 - \partial_1 F_2). \tag{A.15}$$

Appendix B

In order to relate the right-hand side of Eq. (A.15) to $\nabla \times \mathbf{F}$, we need a general expression for $\nabla \times \mathbf{F}$ in arbitrary curvilinear coordinates. Since in the physics literature one almost exclusively uses orthogonal coordinate systems, we include here the result for arbitrary coordinates for convenience. The coordinate independent definition of $\nabla \times \mathbf{F}$ follows from Stokes's theorem, and is [2]

$$\hat{\mathbf{N}} \cdot (\nabla \times \mathbf{F}) = \frac{1}{\Delta A} \oint_C \mathbf{F} \cdot \hat{\boldsymbol{\omega}} \, \mathrm{d}s, \quad \Delta A \to 0. \tag{B.1}$$

Here, C is the boundary curve of the small surface area ΔA, and $\hat{\mathbf{N}}$ is the unit normal vector on ΔA. Vector $\hat{\boldsymbol{\omega}}$ is the unit tangent vector to C, and its direction goes together with $\hat{\mathbf{N}}$ according to the right-hand rule.

In order to evaluate $\nabla\times\mathbf{F}$, we represent a point in space as $\mathbf{r}=\mathbf{r}(u^1,u^2,u^3)$, with u^1, u^2 and u^3 as free parameters. As in Eq. (12) we now have three basis vectors at each point, defined by $\mathbf{c}_i=\partial_i\,\mathbf{r}(u^1,u^2,u^3)$, which are tangent to the parameter curves and which span the three dimensional tangent space. Just as in Section 4 we define the components of the fundamental tensor as $g_{ij}=\mathbf{c}_i\cdot\mathbf{c}_j$, which can now be organized in a 3×3 matrix. With g^{ij} the matrix elements of the inverse matrix we define a reciprocal basis by $\mathbf{c}^i=g^{ij}\mathbf{c}_j$, for which $\mathbf{c}^i\cdot\mathbf{c}_k=\delta^i_k$. Any vector $\mathbf{T}$ in the tangent space can then be written as $\mathbf{T}=T^i\mathbf{c}_i$ or $\mathbf{T}=T_i\mathbf{c}^i$, and $T^i=\mathbf{c}^i\cdot\mathbf{T}$, $T_i=\mathbf{c}_i\cdot\mathbf{T}$. The reciprocal basis vectors $\mathbf{c}^i$ can be constructed explicitly. We have

$$\mathbf{c}^3=\frac{\mathbf{c}_1\times\mathbf{c}_2}{\mathbf{c}_3\cdot(\mathbf{c}_1\times\mathbf{c}_2)},\tag{B.2}$$

and the others follow by cyclic permutation. The equivalent of Eq. (22) becomes

$$\det G=|\mathbf{c}_3\cdot(\mathbf{c}_1\times\mathbf{c}_2)|^2.\tag{B.3}$$

The evaluation of $\nabla\times\mathbf{F}$ follows the same steps as the derivation of $\nabla_S\cdot\mathbf{T}$ in Section 6. We consider the area ΔA as part of the parameter plane $u^3=$ constant, as shown in Fig. 5. We take

$$\hat{\mathbf{N}}=\frac{1}{|\mathbf{c}_1\times\mathbf{c}_2|}\mathbf{c}_1\times\mathbf{c}_2,\tag{B.4}$$

so that the orientation of C is counterclockwise in the figure. Comparison with Eq. (B.2) shows

$$\hat{\mathbf{N}}=\frac{1}{|\mathbf{c}_1\times\mathbf{c}_2|}\{\mathbf{c}_3\cdot(\mathbf{c}_1\times\mathbf{c}_2)\}\mathbf{c}^3.\tag{B.5}$$

For the left-hand side of ΔA we have $\hat{\boldsymbol{\omega}}=-\mathbf{c}_2/|\mathbf{c}_2|$ and $\Delta s=|\mathbf{c}_2|\Delta u^2$, so that $\hat{\boldsymbol{\omega}}\Delta s=-\mathbf{c}_2\Delta u^2$ and $\mathbf{F}\cdot\hat{\boldsymbol{\omega}}\Delta s=-F_2\Delta u^2$.

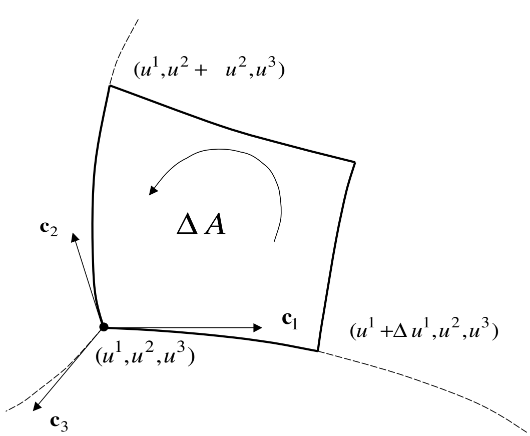

Fig. 5. For the evaluation of $\nabla\times\mathbf{F}$ in Appendix B, we consider the small area ΔA in the plane $u^3=$ constant, as shown in the figure. With the normal vector chosen as in Eq. (B.4), the orientation of the loop is counterclockwise. Vector $\mathbf{c}_3$ is shown for illustration, but it may equally well point towards the other side of ΔA.

For the right-hand side of ΔA everything is the same, except that $\hat{\boldsymbol{\omega}}$ picks up a minus sign and that F_2 has to be evaluated at $(u^1+\Delta u^1,u^2,u^3)$. The sum of the contributions from the left and the right is then $(\partial_1F_2)\Delta u^1\Delta u^2$. For the top and the bottom we get $-(\partial_2F_1)\Delta u^1\Delta u^2$, and therefore

$$\oint_C\mathbf{F}\cdot\hat{\boldsymbol{\omega}}\,\mathrm{d}s\approx(\partial_1F_2-\partial_2F_1)\Delta u^1\Delta u^2.\tag{B.6}$$

The surface area is $\Delta A\approx|\mathbf{c}_1\times\mathbf{c}_2|\Delta u^1\Delta u^2$, so Eq. (B.1) becomes, after substituting $\hat{\mathbf{N}}$ from Eq. (B.5)

$$\mathbf{c}^3\cdot(\nabla\times\mathbf{F})=\frac{1}{\mathbf{c}_3\cdot(\mathbf{c}_1\times\mathbf{c}_2)}(\partial_1F_2-\partial_2F_1),\tag{B.7}$$

which is the final result. The other two components of $\nabla\times\mathbf{F}$ follow by cyclic permutation of the indices. For a right-handed coordinate system, as in Fig. 5, we have $\mathbf{c}_3\cdot(\mathbf{c}_1\times\mathbf{c}_2)>0$, and Eq. (B.7) can be simplified a little by setting $\mathbf{c}_3\cdot(\mathbf{c}_1\times\mathbf{c}_2)=\sqrt{\det G}$, according to Eq. (B.3).

Appendix C

We now consider again the surface S of the interface between the two media, which is parametrized by u^1 and u^2. We then add a third coordinate u^3 to represent a point off the surface S, in such a way that, for instance, $u^3=0$ corresponds to a point on S. This gives a parametrization of space around S with u^1, u^2 and u^3 as free parameters. Alternatively, let space be parametrized with u^1, u^2 and u^3, as in Appendix B. Then each $u^3=$ constant represents a parameter surface in space. Then we assume that the parametrization is chosen such that for one particular value of u^3 the parameter surface coincides with S.

For each point on S the unit normal vector $\hat{\mathbf{n}}$ is given by Eq. (14), and since we now have a third parameter u^3, we also have the reciprocal vector $\mathbf{c}^3$, given by Eq. (B.2). No matter how we choose this third parameter, both vectors are related as

$$\mathbf{c}^3=\frac{\pm1}{\mathbf{c}_3\cdot(\mathbf{c}_1\times\mathbf{c}_2)}|\mathbf{c}_1\times\mathbf{c}_2|\hat{\mathbf{n}}.\tag{C.1}$$

Then we substitute this expression for $\mathbf{c}^3$ in the right-hand side of Eq. (B.7), which yields

$$|\mathbf{c}_1\times\mathbf{c}_2|\hat{\mathbf{n}}\cdot(\nabla\times\mathbf{F})=\pm(\partial_1F_2-\partial_2F_1),\tag{C.2}$$

and with Eq. (22) this is

$$\hat{\mathbf{n}}\cdot(\nabla\times\mathbf{F})=\frac{\pm1}{\sqrt{\det G}}(\partial_1F_2-\partial_2F_1),\tag{C.3}$$

with G the G of the surface. The right-hand side of Eq. (C.3) is just the negative of the right-hand side of Eq. (A.15), so that

$$\nabla_S\cdot(\hat{\mathbf{n}}\times\mathbf{F})=-\hat{\mathbf{n}}\cdot(\nabla\times\mathbf{F}),\tag{C.4}$$

which is the theorem used in Section 7.

With the third parameter u^3 introduced to represent points in space in the neighborhood of S, Eq. (14) defines an $\hat{\mathbf{n}}$ for each point in space covered by the parametrization, so this makes $\hat{\mathbf{n}}$ a vector field in space. This $\hat{\mathbf{n}}$ is perpendicular to the parameter surface $u^3 = \text{constant}$ through that point. With $\mathbf{F}$ another vector field in space, we have the vector identity $\nabla \cdot (\hat{\mathbf{n}} \times \mathbf{F}) = \mathbf{F} \cdot (\nabla \times \hat{\mathbf{n}}) - \hat{\mathbf{n}} \cdot (\nabla \times \mathbf{F})$, where the divergence on the left-hand side is the regular divergence in space. Here we have the additional term $\mathbf{F} \cdot (\nabla \times \hat{\mathbf{n}})$ on the right-hand side, as compared to Eq. (C.4).

Since the vector $\hat{\mathbf{n}}$, defined by Eq. (14), is now a vector field in space, we can compute $\nabla \times \hat{\mathbf{n}}$. As shown in Appendix B, for the curl we need the covariant components of $\hat{\mathbf{n}}$, which are $n_i = \mathbf{c}_i \cdot \hat{\mathbf{n}}$. From Eq. (14) these are $n_1 = n_2 = 0$, and

$$n_3 = \frac{\pm 1}{|\mathbf{c}_1 \times \mathbf{c}_2|} \mathbf{c}_3 \cdot (\mathbf{c}_1 \times \mathbf{c}_2). \tag{C.5}$$

From Eq. (B.7) and the cyclic permutations for the other components, we can then evaluate $\nabla \times \hat{\mathbf{n}}$. With some very serious effort we find

$$\begin{aligned} \nabla \times \hat{\mathbf{n}} = {} & \frac{\pm 1}{|\mathbf{c}_1 \times \mathbf{c}_2|} \mathbf{c}^3 \times \partial_3 (\mathbf{c}_1 \times \mathbf{c}_2) \\ & \pm \frac{1}{|\mathbf{c}_1 \times \mathbf{c}_2|^3} \mathbf{c}^3 \times \{[(\mathbf{c}_1 \times \mathbf{c}_2) \cdot (\mathbf{c}_2 \times \mathbf{c}_3)] \partial_1 (\mathbf{c}_1 \times \mathbf{c}_2) \\ & + [(\mathbf{c}_1 \times \mathbf{c}_2) \cdot (\mathbf{c}_3 \times \mathbf{c}_1)] \partial_2 (\mathbf{c}_1 \times \mathbf{c}_2)\} \end{aligned} \tag{C.6}$$

for $\nabla \times \hat{\mathbf{n}}$ in arbitrary curvilinear coordinates. The partial derivatives of $\mathbf{c}_1 \times \mathbf{c}_2$ which enter this expression are all crossed with $\mathbf{c}^3$, and we have explicitly

$$\mathbf{c}^3 \times \partial_j (\mathbf{c}_1 \times \mathbf{c}_2) = \begin{Bmatrix} 3 \\ j \quad 2 \end{Bmatrix} \mathbf{c}_1 - \begin{Bmatrix} 3 \\ j \quad 1 \end{Bmatrix} \mathbf{c}_2. \tag{C.7}$$

Here,

$$\begin{Bmatrix} k \\ j \quad i \end{Bmatrix} = \mathbf{c}^k \cdot \partial_j \mathbf{c}_i \tag{C.8}$$

are the Christoffel symbols for the parametrization of space with curvilinear coordinates u^1, u^2 and u^3, as in Appendix B. We now see that in Eq. (C.6) each term on the right-hand side is a linear combination of $\mathbf{c}_1$ and $\mathbf{c}_2$, so $\nabla \times \hat{\mathbf{n}}$ is in the tangent plane to the surface $u^3 = \text{constant}$, which is spanned by $\mathbf{c}_1$ and $\mathbf{c}_2$, from which we have $\hat{\mathbf{n}} \cdot (\nabla \times \hat{\mathbf{n}}) = 0$. If at least one of the Christoffel symbols appearing in Eq. (C.7) is nonzero, we could have $\nabla \times \hat{\mathbf{n}} \neq 0$. In that case, $\nabla_S \cdot (\hat{\mathbf{n}} \times \mathbf{F})$ is not equal to $\nabla \cdot (\hat{\mathbf{n}} \times \mathbf{F})$, which has the additional term $\mathbf{F} \cdot (\nabla \times \hat{\mathbf{n}})$.

For an orthogonal system, where the vectors $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$ are mutually perpendicular, Eq. (C.6) simplifies to

$$\nabla \times \hat{\mathbf{n}} = \frac{\pm 1}{|\mathbf{c}_1 \times \mathbf{c}_2|} \mathbf{c}^3 \times \partial_3 (\mathbf{c}_1 \times \mathbf{c}_2), \tag{C.9}$$

which is interesting in its own right.

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