Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Optics Communications 252 (2005) 253–261
OPTICS COMMUNICATIONS
www.elsevier.com/locate/optcom

Vortices in multipole radiation

Henk F. Arnoldus *

Department of Physics and Astronomy, Mississippi State University, P.O. Drawer 5167, Mississippi, MS 39762-5167, USA

Received 27 January 2005; received in revised form 12 April 2005; accepted 14 April 2005

Abstract

The Poynting vector for electric and magnetic multipole radiation of arbitrary order (ℓ, m) has been obtained, and an expression for the field lines of this vector field has been derived. It is shown that the field lines lie on a cone, and that, for $m \neq 0$, they exhibit a vortex structure with a dimension of about a wavelength around the multipole. The field lines wind around the z-axis in the neighborhood of the multipole, and outside this vortex region the field lines run approximately radially outward. We have also derived the asymptotic limit of a field line.

PACS: 03.50.De; 42.25.Bs

Keywords: Multipole radiation; Poynting vector; Optical vortex

1. Introduction

The energy flow in an electromagnetic field is represented by the field lines of the Poynting vector $\mathbf{S}(\mathbf{r})$. For a time-harmonic field with angular frequency ω this vector field is time independent and is given by

$$\mathbf{S}(\mathbf{r}) = \frac{1}{2\mu_0} \mathrm{Re}\, \mathbf{E}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})^*, \tag{1}$$

with $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ the complex amplitudes of the electric and magnetic fields, respectively. Most interesting are radiation patterns which exhibit a vortex structure in which the field lines of $\mathbf{S}(\mathbf{r})$ swirl around a singular point or a singular line. Such vortices occur quite naturally in radiation fields. The oldest example is diffraction of a plane wave by a half-infinite screen where a vortex appears at the illuminated side of the screen [1]. In this case the singularity is a line which runs parallel to the screen, and the field lines of the Poynting vector curl around this line. Vortices also appear in the diffracted field by a slit in a screen [2,3], in the interference pattern of three plane waves [4], in Laguerre–Gaussian laser beams [5–8], and in the focal plane of a lens [9,10]. For non-stationary fields the singular points or lines may propagate [11].

* Tel.: +1 6623 252 919; fax: +1 6623 258 898.
E-mail address: arnoldus@ra.msstate.edu.

0030-4018/$ - see front matter
doi:10.1016/j.optcom.2005.04.033

In the mentioned examples, the vortices are typically a result of diffraction or interference. At a singular point or line the Poynting vector vanishes, and the direction of $\mathbf{S}(\mathbf{r})$ in the neighborhood of the singularity is undetermined. In the case of the Laguerre–Gaussian beam, the rotation in the vortex is an indication of the angular momentum that is carried by the laser beam. It was shown recently [12] that a vortex is present in the field of an electrical dipole when the radiation is emitted in a $\Delta m = \pm 1$ atomic transition. In such a transition, the dipole moment rotates in the xy-plane if the z-axis is taken as the quantization axis. The field lines of the Poynting vector circle around the z-axis with the same orientation as the rotation of the dipole moment. The field lines emanate from the location of the dipole ($\mathbf{r} = 0$), making the origin of coordinates a singular point. In contrast to the previous examples, the magnitude of the Poynting vector near the singular point grows without limits. Another difference is that the rotation in the emitted field is a reflection of the rotation of the source, rather than a result of interference. In this Communication we show that a vortex appears in the Poynting vector of electric and magnetic multipole radiation of arbitrary order, and we derive an explicit formula for the field lines.

2. Poynting vector of a multipole field

We shall consider a source of multipole radiation located at the origin of coordinates, and embedded in a medium with index of refraction n (assumed to be positive). The order of the multipole is indicated by (ℓ, m), with $\ell = 1$ for a dipole, $\ell = 2$ for a quadrupole, etc., and for a given ℓ the value of m can be $-\ell, -\ell+1, \ldots, \ell$. The type of multipole will be distinguished by the parameter η, and we use $\eta = 1$ and $\eta = -1$ for a magnetic and an electric multipole, respectively. The electric and magnetic fields emitted by a multipole of type η and order (ℓ, m) can be written in the compact form

$$\mathbf{E}(\mathbf{r}) = \frac{\mathrm{i}k_0^3}{4\pi\varepsilon_0} b_{\eta\ell m}\mathbf{A}_{\eta\ell m}(\mathbf{r}), \tag{2}$$

$$\mathbf{B}(\mathbf{r}) = \frac{n}{c}\frac{\mathrm{i}k_0^3}{4\pi\varepsilon_0}\eta b_{\eta\ell m}\mathbf{A}_{-\eta\ell m}(\mathbf{r}), \tag{3}$$

in terms of the standard multipole potentials $\mathbf{A}_{\eta\ell m}(\mathbf{r})$ (defined below). Here, $k_0 = \omega/c$, and the overall constant $b_{\eta\ell m}$, the multipole coefficient, is determined by the current density of the source emitting the radiation.

When we substitute expressions (2) and (3) into Eq. (1) we obtain

$$\mathbf{S}_{\eta\ell m}(\mathbf{r}) = n^2k_0^2P_1\eta\,\mathrm{Re}\,\mathbf{A}_{\eta\ell m}(\mathbf{r}) \times \mathbf{A}_{-\eta\ell m}(\mathbf{r})^*, \tag{4}$$

where we have added the subscripts η, ℓ and m to the Poynting vector. The overall constant P_1 is defined by

$$P_1 = \frac{\omega^4}{32\pi^2\varepsilon_0c^3n}|b_{\eta\ell m}|^2, \tag{5}$$

which equals the power emitted by the multipole (Eq. (31) below). Replacing η by $-\eta$ on the right-hand side of Eq. (4) has no effect, and therefore the Poynting vectors for electric and magnetic multipoles of the same order are the same.

The multipole fields are often defined by considering the action of the orbital angular momentum operator $\mathbf{L} = -\mathrm{i}\mathbf{r}\times\nabla$ on the spherical harmonics $Y_{\ell m}(\theta, \phi)$ [13]. A more practical representation [14–17] is in terms of vector spherical harmonics, defined by

$$\mathbf{T}_{\ell\ell' m}(\theta, \phi) = \sum_{m'\mu}(\ell' m' 1\mu\,|\,\ell m)Y_{\ell' m'}(\theta, \phi)\mathbf{e}_\mu, \tag{6}$$

with $(\ell' m' 1\mu|\ell m)$ a Clebsch–Gordan coefficient and $\mathbf{e}_\mu$ a spherical unit vector. The $\eta = 1$ multipole potential is then

$$\mathbf{A}_{1\ell m}(\mathbf{r}) = h_\ell^{(1)}(q)\mathbf{T}_{\ell\ell m}(\theta, \phi), \tag{7}$$

where we have set $q = nk_0r$ for the dimensionless distance between the multipole and the field point $\mathbf{r}$, and $h_\ell^{(1)}(q)$ is a spherical Hankel function. For $\eta = -1$ the multipole potential is

$$\mathbf{A}_{-1\ell m}(\mathbf{r}) = \sqrt{\frac{\ell+1}{2\ell+1}}h_{\ell-1}^{(1)}(q)\mathbf{T}_{\ell\ell-1m}(\theta, \phi) - \sqrt{\frac{\ell}{2\ell+1}}h_{\ell+1}^{(1)}(q)\mathbf{T}_{\ell\ell+1m}(\theta, \phi). \tag{8}$$

3. Evaluation of the Poynting vector

In order to evaluate the Poynting vector explicitly, we substitute the right-hand side of Eq. (7) and the complex conjugate of the right-hand side of Eq. (8) into Eq. (4) with $\eta = 1$. This gives an expression involving three spherical Hankel functions and two cross products between vector spherical harmonics. In Eq. (6), the vector spherical harmonics are expressed in terms of spherical unit vectors $\mathbf{e}_\mu$, $\mu = 1, 0, -1$. For the present calculation it is more convenient to use the spherical-coordinate unit vectors $\hat{\mathbf{r}}$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ for the representation of the functions $\mathbf{T}_{\ell\ell' m}(\theta,\phi)$. These expressions are given in Appendix A.

When we take the cross product of Eqs. (A.1) and (A.3) with $\hat{\mathbf{r}}$ and compare to Eq. (A.2) we obtains

$$\mathrm{i}\hat{\mathbf{r}} \times \mathbf{T}_{\ell\ell+1m}(\theta,\phi) = -\sqrt{\frac{\ell}{2\ell+1}}\mathbf{T}_{\ell\ell m}(\theta,\phi), \tag{9}$$

$$\mathrm{i}\hat{\mathbf{r}} \times \mathbf{T}_{\ell\ell-1m}(\theta,\phi) = -\sqrt{\frac{\ell+1}{2\ell+1}}\mathbf{T}_{\ell\ell m}(\theta,\phi). \tag{10}$$

Then we take again the cross product with $\hat{\mathbf{r}}$, use a vector identity for $\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \ldots)$, and we use Eqs. (A.4) and (A.6) for the appearing dot products. This yields

$$\mathbf{T}_{\ell\ell+1m}(\theta,\phi) = -\sqrt{\frac{\ell+1}{2\ell+1}}Y_{\ell m}(\theta,\phi)\hat{\mathbf{r}} - \mathrm{i}\sqrt{\frac{\ell}{2\ell+1}}\hat{\mathbf{r}} \times \mathbf{T}_{\ell\ell m}(\theta,\phi), \tag{11}$$

$$\mathbf{T}_{\ell\ell-1m}(\theta,\phi) = \sqrt{\frac{\ell}{2\ell+1}}Y_{\ell m}(\theta,\phi)\hat{\mathbf{r}} - \mathrm{i}\sqrt{\frac{\ell+1}{2\ell+1}}\hat{\mathbf{r}} \times \mathbf{T}_{\ell\ell m}(\theta,\phi). \tag{12}$$

Subsequently we take the cross product of the complex conjugate of these equations with $\mathbf{T}_{\ell\ell m}(\theta,\phi)$. For the last terms on the right-hand sides this leads to

$$\mathbf{T}_{\ell\ell m}(\theta,\phi) \times [\hat{\mathbf{r}} \times \mathbf{T}_{\ell\ell m}(\theta,\phi)^*] = [\mathbf{T}_{\ell\ell m}(\theta,\phi) \times \mathbf{T}_{\ell\ell m}(\theta,\phi)^*]\hat{\mathbf{r}}, \tag{13}$$

where we have used $\hat{\mathbf{r}} \cdot \mathbf{T}_{\ell\ell m}(\theta,\phi) = 0$, Eq. (A.5). From Eq. (A.2) we find

$$\mathbf{T}_{\ell\ell m}(\theta,\phi) \cdot \mathbf{T}_{\ell\ell m}(\theta,\phi)^* = \frac{1}{\ell(\ell+1)}\left[\frac{m^2}{\sin^2\theta}|Y_{\ell m}(\theta,\phi)|^2 + \left|\frac{\partial Y_{\ell m}}{\partial\theta}\right|^2\right]. \tag{14}$$

The ϕ dependence of the spherical harmonics can be written as $Y_{\ell m}(\theta,\phi) = Y_{\ell m}(\theta,0)\exp(im\phi)$ with $Y_{\ell m}(\theta,0)$ a real-valued function of θ. Therefore, $\partial Y_{\ell m}/\partial\theta = \exp(im\phi)\partial Y_{\ell m}(\theta,0)/\partial\theta$, and from this observation it follows that the right-hand side of Eq. (14) is independent of ϕ. We introduce the abbreviation

$$N_{\ell m}(\theta) = \mathbf{T}_{\ell\ell m}(\theta,\phi) \cdot \mathbf{T}_{\ell\ell m}(\theta,\phi)^*, \tag{15}$$

which equals the normalized emitted power per unit solid angle of a multipole of order (ℓ, m) (see below Eq. (32)). We then obtain the desired cross products:

$$\mathbf{T}_{\ell\ell m}(\theta,\phi) \times \mathbf{T}_{\ell\ell+1m}(\theta,\phi)^* = \mathrm{i}\sqrt{\frac{\ell}{2\ell+1}}N_{\ell m}(\theta)\hat{\mathbf{r}} - \sqrt{\frac{\ell+1}{2\ell+1}}Y_{\ell m}(\theta,\phi)^* \times \mathbf{T}_{\ell\ell m}(\theta,\phi) \times \hat{\mathbf{r}}, \tag{16}$$

$$\mathbf{T}_{\ell\ell m}(\theta,\phi) \times \mathbf{T}_{\ell\ell-1m}(\theta,\phi)^* = \mathrm{i}\sqrt{\frac{\ell+1}{2\ell+1}}N_{\ell m}(\theta)\hat{\mathbf{r}} + \sqrt{\frac{\ell}{2\ell+1}}Y_{\ell m}(\theta,\phi)^* \times \mathbf{T}_{\ell\ell m}(\theta,\phi) \times \hat{\mathbf{r}}. \tag{17}$$

For the Poynting vector we need the cross product in Eq. (4) with $\eta = 1$, and this now becomes

$$\mathbf{A}_{1\ell m}(\mathbf{r}) \times \mathbf{A}_{-1\ell m}(\mathbf{r})^* = \frac{\mathrm{i}}{2\ell+1}h_\ell^{(1)}(q)\left[(\ell+1)h_{\ell-1}^{(1)}(q)^* - \ell h_{\ell+1}^{(1)}(q)^*\right] \times N_{\ell m}(\theta)\hat{\mathbf{r}} + \frac{\sqrt{\ell(\ell+1)}}{2\ell+1}h_\ell^{(1)}(q)\left[h_{\ell-1}^{(1)}(q)^* + h_{\ell+1}^{(1)}(q)^*\right]Y_{\ell m}(\theta,\phi)^*\mathbf{T}_{\ell\ell m}(\theta,\phi) \times \hat{\mathbf{r}}. \tag{18}$$

The second term on the right-hand side can be simplified with a recurrence relation for spherical Bessel functions:

$$h_{\ell-1}^{(1)}(q) + h_{\ell+1}^{(1)}(q) = \frac{2\ell+1}{q}h_\ell^{(1)}(q). \tag{19}$$

For the Poynting vector we need the real part of the right-hand side of Eq. (18). One of the

Wronski relations for spherical Bessel functions [18] can be written as

$$\mathrm{Re}\, \mathrm{i} h_\ell^{(1)}(q) h_{\ell\pm 1}^{(1)}(q)^* = \mp \frac{1}{q^2}, \tag{20}$$

and this gives

$$\mathrm{Re}\, \mathbf{A}_{1\ell m}(\mathbf{r}) \times \mathbf{A}_{-1\ell m}(\mathbf{r})^* = \frac{1}{q^2} N_{\ell m}(\theta)\hat{\mathbf{r}} - \sqrt{\ell(\ell+1)}\frac{1}{q}|h_\ell^{(1)}(q)|^2 \times \hat{\mathbf{r}} \times \mathrm{Re}\, Y_{\ell m}(\theta,\phi)^* \mathbf{T}_{\ell\ell m}(\theta,\phi). \tag{21}$$

When we multiply Eq. (A.2) by $Y_{\ell m}(\theta,\phi)^*$, then the second term on the right-hand side is pure imaginary, and gives no contribution. We obtain

$$\mathrm{Re}\, Y_{\ell m}(\theta,\phi)^* \mathbf{T}_{\ell\ell m}(\theta,\phi) = -\frac{1}{\sqrt{\ell(\ell+1)}} \frac{m}{\sin\theta} \mathbf{e}_\theta |Y_{\ell m}(\theta,\phi)|^2, \tag{22}$$

which simplifies the right-hand side of Eq. (21). We now introduce the abbreviation

$$M_{\ell m}(\theta) = \frac{1}{\sin\theta}|Y_{\ell m}(\theta,\phi)|^2, \tag{23}$$

which is independent of ϕ. The remaining spherical Hankel function in Eq. (21) is given by [19]

$$h_\ell^{(1)}(q) = (-\mathrm{i})^{\ell+1}\frac{\mathrm{e}^{\mathrm{i}q}}{q}\sum_{k=0}^{\ell}\frac{(\ell+k)!}{k!(\ell-k)!}\left(\frac{\mathrm{i}}{2q}\right)^k, \tag{24}$$

so if we set

$$A_\ell(q) = \left|\sum_{k=0}^{\ell}\frac{(\ell+k)!}{k!(\ell-k)!}\left(\frac{\mathrm{i}}{2q}\right)^k\right|^2, \tag{25}$$

then Eq. (21) becomes

$$\mathrm{Re}\, \mathbf{A}_{1\ell m}(\mathbf{r}) \times \mathbf{A}_{-1\ell m}(\mathbf{r})^* = \frac{1}{q^2} N_{\ell m}(\theta)\hat{\mathbf{r}} + \frac{m}{q^3} A_\ell(q) M_{\ell m}(\theta)\mathbf{e}_\phi. \tag{26}$$

Finally, the Poynting vector from Eq. (4) becomes

$$\mathbf{S}_{\eta\ell m}(\mathbf{r}) = \frac{P_1}{r^2}\left[N_{\ell m}(\theta)\hat{\mathbf{r}} + \frac{m}{q}A_\ell(q)M_{\ell m}(\theta)\mathbf{e}_\phi\right]. \tag{27}$$

The result (27) is remarkably simple in appearance. The function $N_{\ell m}(\theta)$, which equals the right-hand side of Eq. (14), involves the derivative of $Y_{\ell m}(\theta,\phi)$ with respect to θ. With recursion relations for spherical harmonics [20], the function $N_{\ell m}(\theta)$ can be cast in the alternative form

$$N_{\ell m}(\theta) = \frac{1}{\ell(\ell+1)}\left[m^2|Y_{\ell m}(\theta,\phi)|^2 + \frac{1}{2}(\ell+m) \times (\ell-m+1)|Y_{\ell m-1}(\theta,\phi)|^2 + \frac{1}{2}(\ell-m) \times (\ell+m+1)|Y_{\ell m+1}(\theta,\phi)|^2\right]. \tag{28}$$

Therefore, both $N_{\ell m}(\theta)$ and $M_{\ell m}(\theta)$ are determined by the absolute values of spherical harmonics. These functions are independent of ϕ, and from the properties of spherical harmonics it also follows that these functions are independent of the sign of m. These functions are listed in Appendix A for $\ell = 1$ and $\ell = 2$. The dependence on the dimensionless radial distance q between the multipole and the field point enters through the function $A_\ell(q)$, which is a polynomial in $1/q^2$ with leading term $A_\ell(q) = 1 + \mathcal{O}(q^{-2})$. The Poynting vector $\mathbf{S}_{\eta\ell m}(\mathbf{r})$ has a radial part, proportional to $\hat{\mathbf{r}}$, and a term proportional to $\mathbf{e}_\phi$, indicating a rotation around the z-axis. The sign of m only enters through the factor m/q, and therefore this rotation is positive or negative with the sign of m (the positive direction being the direction which follows with the right-hand rule from the orientation of the z-axis).

In general, the power passing through a surface element of a sphere with radius r around the origin and with solid angle $\mathrm{d}\Omega$ is given by

$$\mathrm{d}P_{\eta\ell m} = \mathbf{S}_{\eta\ell m}(\mathbf{r}) \cdot \hat{\mathbf{r}} r^2 \,\mathrm{d}\Omega. \tag{29}$$

For multipole radiation this becomes with Eq. (27)

$$\frac{\mathrm{d}P_{\eta\ell m}}{\mathrm{d}\Omega} = P_1 N_{\ell m}(\theta), \tag{30}$$

which is the power per unit solid angle. We note that the dependence on r cancels. The total emitted power by the multipole is then

$$\int \mathrm{d}\Omega \frac{\mathrm{d}P_{\eta\ell m}}{\mathrm{d}\Omega} = P_1, \tag{31}$$

since

$$\int \mathrm{d}\Omega\, N_{\ell m}(\theta) = 1, \tag{32}$$

which in turn follows from the fact that the vector spherical harmonics $\mathbf{T}_{\ell\ell m}(\theta,\phi)$ are normalized on the unit sphere. Therefore, the parameter P_1 is the total power, and the function $N_{\ell m}(\theta)$ is the normalized power per unit solid angle.

4. Field lines of the Poynting vector

The term with $\mathbf{e}_\phi$ in Eq. (27) does not affect the power per unit solid angle, but it does determine the direction of energy flow out of the multipole. To see this, we now determine the field lines of the Poynting vector. Let $\mathbf{r}(u)$, with u a dummy parameter, represent a field line. For any point on such a line, the Poynting vector at that point must be on the tangent line. Field lines are only determined by the direction of $\mathbf{S}_{\eta\ell m}(\mathbf{r})$ at $\mathbf{r}$, and therefore the vector field $f(\mathbf{r})\mathbf{S}_{\eta\ell m}(\mathbf{r})$, with $f(\mathbf{r})$ an arbitrary positive function of $\mathbf{r}$, has the same field lines. Consequently, field lines are the solution of the autonomous differential equation

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u} = f(\mathbf{r})\mathbf{S}_{\eta\ell m}(\mathbf{r}). \tag{33}$$

With $1/(nk_0)$ as the unit of length, a field point is represented by the dimensionless vector $\mathbf{q} = nk_0\mathbf{r}$. A convenient choice for the function $f(\mathbf{r})$ is

$$f(\mathbf{r}) = \frac{r^2}{nk_0 P_1 N_{\ell m}(\theta)}. \tag{34}$$

With Eq. (27), the equation for the field lines then becomes

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}u} = \hat{\mathbf{q}} + \frac{m}{q}A_\ell(q)\frac{M_{\ell m}(\theta)}{N_{\ell m}(\theta)}\mathbf{e}_\phi, \tag{35}$$

with $\hat{\mathbf{q}} = \hat{\mathbf{r}}$.

The spherical coordinates of $\mathbf{q}$ are (q, θ, ϕ), and writing out Eq. (35) in terms of these coordinates yields

$$\frac{\mathrm{d}q}{\mathrm{d}u} = 1, \tag{36}$$

$$q\frac{\mathrm{d}\theta}{\mathrm{d}u} = 0, \tag{37}$$

$$q\sin\theta\frac{\mathrm{d}\phi}{\mathrm{d}u} = \frac{m}{q}A_\ell(q)\frac{M_{\ell m}(\theta)}{N_{\ell m}(\theta)}. \tag{38}$$

The simple form of the right-hand side of Eq. (36) is a consequence of the choice of $f(\mathbf{r})$. From this equation it follows that we can set $u = q$, so that q becomes the free parameter for a field line. The zero on the right-hand side of Eq. (37) follows from the fact that the Poynting vector has no $\mathbf{e}_\theta$ component, and the solution of this equation is $\theta = \theta_0$, a constant. Therefore, on a field line the angle θ is the same for every point, and this implies that the field line lies on the cone $\theta = \theta_0$. Since $\theta = \theta_0$ on a field line, we can set $\theta = \theta_0$ in Eq. (38). So when we can set

$$\alpha = \frac{M_{\ell m}(\theta_0)}{\sin\theta_0 N_{\ell m}(\theta_0)}, \tag{39}$$

which is a constant on a field line. Then Eq. (38) becomes

$$\frac{\mathrm{d}\phi}{\mathrm{d}q} = \frac{m\alpha}{q^2}A_\ell(q), \tag{40}$$

with solution

$$\phi(q) = \phi_0 - m\alpha\int_q^\infty \frac{\mathrm{d}t}{t^2}A_\ell(t). \tag{41}$$

Here, ϕ_0 is a constant, which equals $\phi(q \to \infty)$. Eq. (41) is the parameter equation of a field line of the Poynting vector (together with $\theta = \theta_0$).

The function $A_\ell(q)$ is a positive function of q, as follows from its definition (25). For a dipole ($\ell = 1$) and a quadrupole ($\ell = 2$) these functions are explicitly

$$A_1(q) = 1 + \frac{1}{q^2}, \tag{42}$$

$$A_2(q) = 1 + \frac{3}{q^2} + \frac{9}{q^4}. \tag{43}$$

The integral in Eq. (41) is therefore a decreasing function of q, which implies that $\phi(q)$ increases (decreases) with q for m positive (negative). A field line of the Poynting vector is therefore a curve which starts at the origin of coordinates ($q = 0$) and which spirals around the z-axis while staying on the cone $\theta = \theta_0$. With increasing q, the distance to the origin increases, and therefore the field line spirals outward. The orientation of the field line is positive and negative with the sign of m. For $m = 0$ the equation for the field line reduces to $\phi(q) = \phi_0$, and this corresponds to a straight field line which runs radially outward. For q large, the coordinates (θ, ϕ) of a point on the field line approach (θ_0, ϕ_0), and in particular $\phi(q)$ becomes approximately constant with increasing q. This means that for large q the spiraling behavior disappears, and the field line runs in the direction (θ_0, ϕ_0) up to infinity. Figs. 1 and 2 show typical field lines. The vortex structure extends for about a wavelength around the site of the multipole, and for larger distances the field lines run approximately radially outward.

The result (41) gives the field line of the Poynting vector for any multipole, and therefore the resulting vortex shown in Figs. 1 and 2 is a universal feature of multipole radiation. The field lines for electric and magnetic multipoles of the same order are the same. The dependence on the order (ℓ, m) enters through the function $A_\ell(q)$ and the parameter α, and this has no influence on the structure of the resulting vortex. Only the explicit appearance of m on the right-hand side of Eq. (41) has significance in that its sign determines the orientation of the field line.

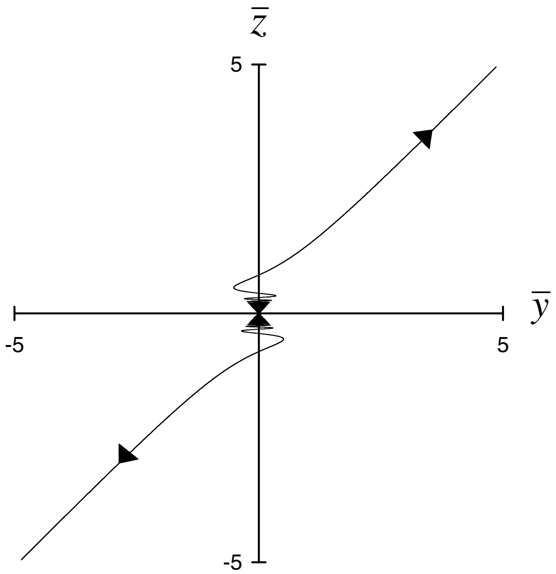

Fig. 1. Illustration of two field lines for a dipole with $m = 1$. We use dimensionless coordinates $(\bar{x}, \bar{y}, \bar{z})$, as in Section 6. Each field line is determined by (θ_0, ϕ_0), and for the two lines shown we took $(\theta_0, \phi_0) = (\pi/4, \pi/2)$ and $(\theta_0, \phi_0) = (3\pi/4, -\pi/2)$. The parameter equations for field lines in Cartesian coordinates are given by Eqs. (46)–(48), and in the figure the field lines are projected onto the $\bar{y}\bar{z}$-plane. A field line spirals an infinite number of times around the z-axis and the length of the spiraling part is infinite.

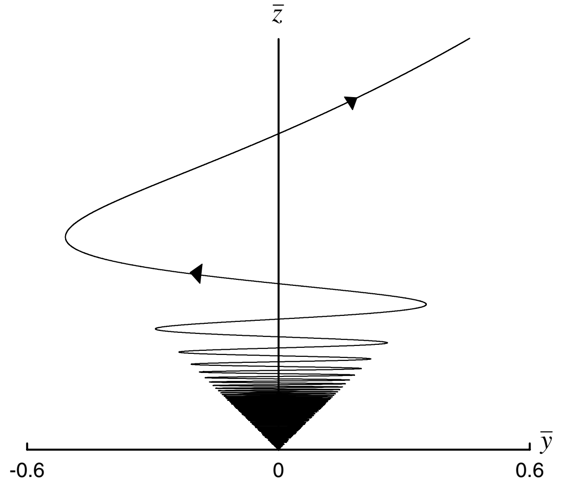

Fig. 2. Enlargement of the field line in $z > 0$ from Fig. 1. Here we see clearly that the field line lies on the cone $\theta_0 = \pi/4$. The arrow to the left (right) is behind (in front of) the $\bar{z}$-axis, corresponding to a positive orientation of the spiraling field line.

5. The z-axis

In most optical vortices the field lines of the Poynting vector curl around a singular point (two-dimensional problems) or a line (three dimensional problems). At such a point or on such a line the Poynting vector vanishes, and in the neighborhood of the point or line the direction (or phase) of the Poynting vector becomes undetermined. For the multipole vortex, the field lines rotate about the z-axis, and one would expect that this makes the z-axis a singular line. We now show that this is not necessarily the case. The dependence on θ enters the Poynting vector, Eq. (27), through the functions $N_{\ell m}(\theta)$ and $M_{\ell m}(\theta)$, and these functions are determined by $Y_{\ell m}(\theta, \phi)$. The spherical harmonics are proportional to $(\sin\theta)^{|m|}$. For a point on the z-axis we have $\theta = 0$ or π, so we have $Y_{\ell m}(\theta, \phi) = 0$ on the z-axis, unless $m = 0$. From Eq. (23) we then observe that $M_{\ell m}(\theta) = 0$ on the z-axis for $m \neq 0$. In Eq. (27), $M_{\ell m}(\theta)$ is multiplied by m, so $mM_{\ell m}(\theta)$ vanishes on the z-axis for all m.

Therefore, the expression for the Poynting vector on the z-axis simplifies to

$$\mathbf{S}_{\eta\ell m}(\mathbf{r}) = \frac{P_1}{r^2} N_{\ell m}(\theta = 0 \text{ or } \pi)\hat{\mathbf{r}}, \tag{44}$$

and $\hat{\mathbf{r}} = \text{sgn}(z)\mathbf{e}_z$. Now we consider the value of $N_{\ell m}(\theta)$ for $\theta = 0$ or π, and we look at expression (28) for this purpose. Since $Y_{\ell m}(\theta, \phi)$ is only non-zero on the z-axis for $m = 0$ we see that $N_{\ell m}(\theta)$ can only be finite on the z-axis for $m = \pm 1$, and this comes from either the second or the third term on the right-hand side of Eq. (28). This yields $N_{\ell \pm 1} = |Y_{\ell 0}|^2/2$, both for $\theta = 0$ and $\theta = \pi$. From the known expressions for spherical harmonics we furthermore have $|Y_{\ell 0}|^2 = (2\ell + 1)/4\pi$ on the z-axis. Therefore we obtain

$$\mathbf{S}_{\eta\ell\pm 1}(\mathbf{r}) = \frac{P_1}{r^2}\frac{2\ell + 1}{8\pi}\hat{\mathbf{r}}, \tag{45}$$

for the Poynting vector on the z-axis, whereas the Poynting vector vanishes identically on the z-axis for other values of m. This shows that for $m = \pm 1$, the positive and negative sides of the z-axis are field lines, on which the Poynting vector is finite. In this case, the z-axis is not a singular line in the usual sense.

6. Asymptotic limit of a field line

A field line is parametrized with the dimensionless distance q between a point on the field line and the origin of coordinates. The spherical-coordinate angles (θ, ϕ) of a point on the field line are $\theta = \theta_0$ and the function $\phi(q)$ is given by Eq. (41). We introduce the dimensionless Cartesian coordinates $\bar{x} = nk_0x$, $\bar{y} = nk_0y$ and $\bar{z} = nk_0z$, so that a field line is represented by the parameter equations

$$\bar{x}(q) = q\sin\theta_0\cos\phi(q), \tag{46}$$

$$\bar{y}(q) = q\sin\theta_0\sin\phi(q), \tag{47}$$

$$\bar{z}(q) = q\cos\theta_0. \tag{48}$$

For $q \to \infty$, the function $\phi(q)$ approaches the value ϕ_0, and with $\phi(q) = \phi_0$ substituted into Eqs. (46)–(48) these equations represent a line through the origin of coordinates with angles (θ_0, ϕ_0). It might therefore seem that a field line with constants (θ_0, ϕ_0) approaches this line for q large. We now show that this is not exactly the case.

The function $A_\ell(t)$ in Eq. (41) is to leading order $A_\ell(t) = 1 + \mathcal{O}(t^{-2})$ for t large. This gives

$$\phi(q) = \phi_0 - \frac{m\alpha}{q} + \mathcal{O}\left(\frac{1}{q^3}\right). \tag{49}$$

With Eqs. (46) and (47) this yields

$$\bar{x}(q) = \sin\theta_0(q\cos\phi_0 + m\alpha\sin\phi_0)[1 + \mathcal{O}(q^{-2})], \tag{50}$$

$$\bar{y}(q) = \sin\theta_0(q\sin\phi_0 - m\alpha\cos\phi_0)[1 + \mathcal{O}(q^{-2})], \tag{51}$$

and the equation for $\bar{z}$ is the same as Eq. (48). For q large, the factors in square brackets vanish, and the field line approaches the line (asymptote) which is parametrized by $\bar{x}(q) = \sin\theta_0(q\cos\phi_0 + m\alpha\sin\phi_0)$, $\bar{y}(q) = \sin\theta_0(q\sin\phi_0 - m\alpha\cos\phi_0)$ and $\bar{z}(q) = q\cos\theta_0$. This is a straight line, but it does not go through the origin of coordinates. For $q = 0$ we have $\bar{z}(0) = 0$, so the value $q = 0$ gives the intersection of the asymptote with the $\bar{x}\bar{y}$-plane. The $\bar{x}$ and $\bar{y}$ values of this point are $\bar{x}(0) = m\alpha\sin\theta_0\sin\phi_0$ and $\bar{y}(0) = -m\alpha\sin\theta_0\cos\phi_0$. Therefore,

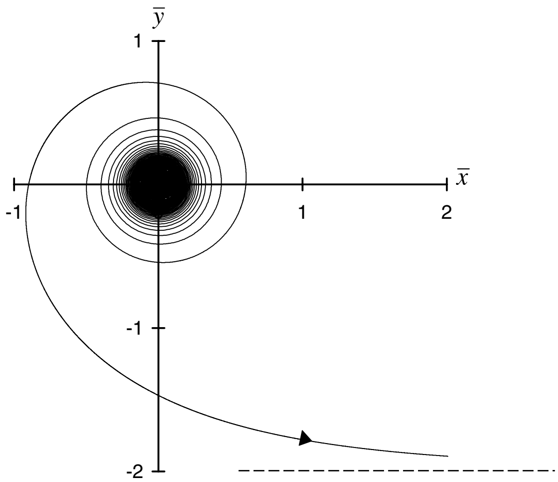

Fig. 3. Shown is a field line in the $\bar{x}\bar{y}$-plane with $\phi_0 = 0$ for a dipole with $m = 1$. The parameter α, defined by Eq. (39), is equal to $\alpha = 2$, as follows from Eqs. (A.7) and (A.9). The dashed line is the asymptote of the field line, and the figure illustrates that the field line approaches the line $\bar{y} = -2$, as predicted by Eq. (52), rather than a line through the origin of coordinates.

for $m \neq 0$ (and $\theta_0 \neq 0$ or π), the asymptote does not go through the origin of coordinates.

This effect is most easily illustrated by considering a field line in the $\bar{x}\bar{y}$-plane, for which $\theta_0 = \pi/2$. The equations for the asymptote become $\bar{x}(q) = q\cos\phi_0 + m\alpha\sin\phi_0$, $\bar{y}(q) = q\sin\phi_0 - m\alpha\cos\phi_0$, and upon eliminating q this is

$$\bar{y} = \bar{x}\tan\phi_0 - \frac{m\alpha}{\cos\phi_0}, \tag{52}$$

a line with $-m\alpha/\cos\phi_0$ as $\bar{y}$-intercept. Fig. 3 shows a field line for a dipole with $m = 1$ and for $\phi_0 = 0$, and the asymptote of this field line.

7. Conclusions

The Poynting vector for radiation emitted by a multipole of arbitrary order (ℓ,m) has been obtained, and the result is given by Eq. (27). Magnetic and electric multipoles have the same Poynting vector. The vector field $\mathbf{S}_{\eta\ell m}(\mathbf{r})$ has a term proportional to $\hat{\mathbf{r}}$, which accounts for the radial outflow of energy, and the term is proportional to $N_{\ell m}(\theta)$, which equals the normalized power per unit solid angle. The second term is proportional to $\mathbf{e}_\phi$, and this term is responsible for the rotation of the field lines around the z-axis. This leads to a vortex structure, as illustrated by the figures. The rotation is positive and negative with the sign of m. For $m = 0$ the $\mathbf{e}_\phi$ component in $\mathbf{S}_{\eta\ell m}(\mathbf{r})$ vanishes, and the field lines are radially outward. The parameter equation for the field lines is given by Eq. (41), in combination with Eqs. (46)–(48) for the Cartesian coordinates of a point on a field line. Each field line is determined by (θ_0, ϕ_0), which equal the asymptotic values of the spherical coordinates (θ, ϕ) of a point on the corresponding field line, but it was shown that the field lines do not approach the line $\theta = \theta_0$, $\phi = \phi_0$ asymptotically. This feature is illustrated in Fig. 3.

Appendix A

Eq. (6) defines the vector spherical harmonics $\mathbf{T}_{\ell\ell'\mathrm{m}}(\theta,\phi)$ in terms of Clebsch–Gordan coefficients, spherical harmonics and spherical unit vectors. The corresponding expressions in terms of spherical-coordinate unit vectors were derived by Hill [21], and here we list the result for reference. The parameter ℓ' can only have the values $\ell + 1$, ℓ and $\ell - 1$, for which the vector spherical harmonics are

$$\mathbf{T}_{\ell\ell+1m}(\theta,\phi) = \frac{1}{\sqrt{(\ell+1)(2\ell+1)}}\left\{\left[\frac{im}{\sin\theta}\mathbf{e}_\phi - (\ell+1)\hat{\mathbf{r}}\right] \times Y_{\ell m}(\theta,\phi) + \mathbf{e}_\theta\frac{\partial Y_{\ell m}}{\partial\theta}\right\}, \tag{A.1}$$

$$\mathbf{T}_{\ell\ell m}(\theta,\phi) = \frac{-1}{\sqrt{\ell(\ell+1)}}\left[\frac{m}{\sin\theta}\mathbf{e}_\theta Y_{\ell m}(\theta,\phi). + i\mathbf{e}_\phi\frac{\partial Y_{\ell m}}{\partial\theta}\right], \tag{A.2}$$

$$\mathbf{T}_{\ell\ell-1m}(\theta,\phi) = \frac{1}{\sqrt{\ell(2\ell+1)}}\left\{\left[\frac{im}{\sin\theta}\mathbf{e}_\phi + \ell\hat{\mathbf{r}}\right]Y_{\ell m}(\theta,\phi) + \mathbf{e}_\theta\frac{\partial Y_{\ell m}}{\partial\theta}\right\}. \tag{A.3}$$

From these expressions we immediately see that

$$\hat{\mathbf{r}}\cdot\mathbf{T}_{\ell\ell+1m}(\theta,\phi) = -\sqrt{\frac{\ell+1}{2\ell+1}}Y_{\ell m}(\theta,\phi), \tag{A.4}$$

$$\hat{\mathbf{r}}\cdot\mathbf{T}_{\ell\ell m}(\theta,\phi) = 0, \tag{A.5}$$

$$\hat{\mathbf{r}}\cdot\mathbf{T}_{\ell\ell-1m}(\theta,\phi) = \sqrt{\frac{\ell}{2\ell+1}}Y_{\ell m}(\theta,\phi). \tag{A.6}$$

Eq. (A.5) expresses that the vector field $\mathbf{A}_{1\ell m}(\mathbf{r})$ is transverse (perpendicular to $\hat{\mathbf{r}}$) for all $\mathbf{r}$.

The Poynting vector $\mathbf{S}_{\eta\ell m}(\mathbf{r})$ depends on θ through the functions $N_{\ell m}(\theta)$ and $M_{\ell m}(\theta)$, given by Eqs. (23) and (28) in terms of spherical harmonics. For reference we list these functions here for a dipole ($\ell = 1$) and a quadrupole ($\ell = 2$). For a dipole we have

$$N_{1\pm1}(\theta) = \frac{3}{8\pi}\left(1 - \frac{1}{2}\sin^2\theta\right), \tag{A.7}$$

$$N_{10}(\theta) = \frac{3}{8\pi}\sin^2\theta, \tag{A.8}$$

$$M_{1\pm1}(\theta) = \frac{3}{8\pi}\sin\theta, \tag{A.9}$$

and for a quadrupole these functions are

$$N_{2\pm2}(\theta) = \frac{5}{16\pi}(1 - \cos^4\theta), \tag{A.10}$$

$$N_{2\pm1}(\theta) = \frac{5}{16\pi}(1 - 3\cos^2\theta + 4\cos^4\theta), \tag{A.11}$$

$$N_{20}(\theta) = \frac{15}{32\pi}(\sin 2\theta)^2, \tag{A.12}$$

$$M_{2\pm2}(\theta) = \frac{15}{32\pi}\sin^3\theta, \tag{A.13}$$

$$M_{2\pm1}(\theta) = \frac{15}{8\pi}\sin\theta\cos^2\theta. \tag{A.14}$$

The functions $M_{\ell m}(\theta)$ are not needed for $m = 0$, since they are multiplied by m in Eq. (27).

References

[1] W. Braunbek, G. Laukien, Optik 9 (1952) 174.
[2] H.F. Schouten, T.D. Visser, G. Gbur, D. Lenstra, H. Blok, Opt. Exp. 11 (2003) 371.
[3] H.F. Schouten, T.D. Visser, G. Gbur, D. Lenstra, H. Blok, Phys. Rev. E 67 (2003) 036608-1-3.
[4] J. Masajada, B. Dubik, Opt. Commun. 198 (2001) 21.
[5] M. Vasnetsov, K. Staliunas (Eds.), Optical Vortices, Horizons in World Physics, Nova Science, vol. 228, Commack, New York, 1999.
[6] A.V. Volyar, V.G. Shvedov, T.A. Fadeeva, Opt. Spectrosc. 90 (2001) 93.
[7] V.A. Pas'ko, M.S. Soskin, M.V. Vasnetsov, Opt. Commun. 198 (2001) 49.
[8] A.V. Volyar, T.A. Fadeeva, V.G. Shvedov, Opt. Spectrosc. 93 (2002) 285.
[9] B. Richards, E. Wolf, Proc. R. Soc. Lond. A 253 (1959) 358.
[10] A. Boivin, J. Dow, E. Wolf, J. Opt. Soc. Am. 57 (1967) 1171.
[11] I. Bialynicki-Birula, Z. Bialynicki-Birula, Phys. Rev. A 67 (2003) 062114.
[12] H.F. Arnoldus, J.T. Foley, Opt. Commun. 231 (2004) 115.
[13] J.D. Jackson, Classical Electrodynamics, third ed., Wiley, New York, 1998, sec. 9.7.
[14] M.E. Rose, Multipole Fields, Wiley, New York, 1955 (Chapter 2).
[15] J.M. Eisenberg, W. Greiner, Excitation Mechanisms of the Nucleus, North-Holland, Amsterdam, 1970 (Chapter 3).
[16] J.M. Blatt, V.F. Weisskopf, Theoretical Nuclear Physics, Dover, New York, 1991. App. B.
[17] M.E. Rose, Elementary Theory of Angular Momentum, Dover, New York, 1995 (Chapter 7).
[18] G.B. Arfken, H.J. Weber, Mathematical Methods for Physicists, fourth ed., Academic Press, San Diego, 1995, p. 663.
[19] M. Abramowitz, I. Stegun, Handbook of Mathematical Functions, Dover, New York, 1970, p. 439.
[20] E.L. Hill, R. Landshoff, Rev. Mod. Phys. 10 (1938) 87.
[21] E.L. Hill, Am. J. Phys. 22 (1954) 211.