JOURNAL OF MODERN OPTICS, 2003, VOL. 50, NO. 5, 755–770

Transverse and longitudinal components of the optical self-, near-, middle- and far-field

HENK F. ARNOLDUS

Department of Physics and Astronomy, Mississippi State University, PO Drawer 5167, Mississippi State, Mississippi, 39762-5167, USA

(*Received 10 November 2001; revision received 5 February 2002*)

Abstract. The electric field emitted by a localized current density has four distinct parts, which are referred to as the self-, near-, middle- and far-field, and each of these parts consists of a longitudinal and a transverse component. We have studied this eight-fold splitting of the field by means of the corresponding dyadic Green's functions, both in configuration space and in reciprocal space. It is shown that each component can be expressed in terms of rather simple universal auxiliary functions.

1. Introduction

Electromagnetic fields emitted by localized sources of atomic dimensions are usually observed in the far-field region, a macroscopic distance away from the radiator. With recent developments in near-field technology, using very small optical fibre tips, it has become experimentally feasible to measure electromagnetic fields in the vicinity of an atom, as close as a few wavelengths distance from the atom or any other microscopic source [1–4]. The electric field splits naturally into four distinct parts: the self-field, near-field, middle-field and the far-field, each with its own specific characteristics. Usually, only the far-field part is considered since this field relates to macroscopic detection of the radiation. However, with the increasing experimental interest in near-field optics it has become necessary to study the other three parts of the field in more detail. It has been shown recently [5–7] that especially the self-field plays a crucial role in radiation phenomena at short distances. Historically, this self-field has been ignored completely since it only exists inside the source [8].

A different kind of splitting of the electric field is into its transverse and longitudinal components. These components could be characterized as the radiating and the attached components of the field, respectively, which is especially evident when one considers the quantization of the field in the Coulomb gauge [9].

In this paper we study the combined splitting of the electric field. The four parts of the field each have transverse and longitudinal components, which can be evaluated explicitly. We shall obtain these eight contributions both in coordinate ($\mathbf{r}$) space and reciprocal ($\mathbf{k}$) space. Rather than splitting the field itself, we consider the dyadic Green's function which relates the field to its source, and we shall split this Green's function directly. In this way, our results hold for any electric field, no matter what its source is.

Journal of Modern Optics ISSN 0950–0340 print/ISSN 1362–3044 online
http://www.tandf.co.uk/journals
DOI: 10.1080/09500340210149732

2. Dyadic Green's function

For a given localized current density $\mathbf{j}(\mathbf{r})$ the electric field is given by

$$\mathbf{E}(\mathbf{r}) = \frac{\mathrm{i}\omega\mu_o}{4\pi}\int \mathrm{d}^3\mathbf{r}' g(\mathbf{r}-\mathbf{r}')\mathbf{j}(\mathbf{r}') + \frac{\mathrm{i}\omega\mu_o}{4\pi k_o^2}\nabla\left(\nabla\cdot\int \mathrm{d}^3\mathbf{r}' g(\mathbf{r}-\mathbf{r}')\mathbf{j}(\mathbf{r}')\right), \tag{1}$$

as follows from Maxwell's equations. We shall assume a harmonic time dependence with angular frequency ω, and suppress the ω dependence of the various quantities in the notation. Here, the scalar Green's function is given by

$$g(\mathbf{r}) = \frac{\exp(\mathrm{i}k_o r)}{r}, \tag{2}$$

with $k_o = \omega/c$. In order to express solution (1) in terms of a Green's function, we need to move the differential operators in the second term on the right-hand side under the integral sign. Due to the singularity in $g(\mathbf{r})$ this yields an additional term [10, 11], and we obtain

$$\nabla\left(\nabla\cdot\int \mathrm{d}^3\mathbf{r}' g(\mathbf{r}-\mathbf{r}')\mathbf{j}(\mathbf{r}')\right) = -\frac{4\pi}{3}\mathbf{j}(\mathbf{r}) + \int \mathrm{d}^3\mathbf{r}'\nabla[\nabla\cdot(g(\mathbf{r}-\mathbf{r}')\mathbf{j}(\mathbf{r}'))]. \tag{3}$$

With some rearrangements, the solution for $\mathbf{E}(\mathbf{r})$ can then be written as

$$\mathbf{E}(\mathbf{r}) = \frac{\mathrm{i}\omega\mu_o}{4\pi}\int \mathrm{d}^3\mathbf{r}'\mathbf{g}(\mathbf{r}-\mathbf{r}')\cdot\mathbf{j}(\mathbf{r}'), \tag{4}$$

with the dyadic Green's function (tensor) $\mathbf{g}(\mathbf{r})$ defined by

$$\mathbf{g}(\mathbf{r}) = -\frac{4\pi}{3k_o^2}\boldsymbol{\delta}(\mathbf{r}) + \left(\mathbf{I} + \frac{1}{k_o^2}\nabla\nabla\right)g(\mathbf{r}), \tag{5}$$

and here $\boldsymbol{\delta}(\mathbf{r}) = \delta(\mathbf{r})\mathbf{I}$ with $\mathbf{I}$ the unit dyad.

Working out the derivatives $\nabla\nabla g(\mathbf{r})$ in (5), and grouping the resulting terms with respect to their r dependence, shows that the dyadic Green's function $\mathbf{g}(\mathbf{r})$ is the sum of the following four parts:

$$\mathbf{g}(\mathbf{r})_{\mathrm{SF}} = -\frac{4\pi}{3k_o^2}\boldsymbol{\delta}(\mathbf{r}), \tag{6}$$

$$\mathbf{g}(\mathbf{r})_{\mathrm{NF}} = -\frac{1}{k_o^2 r^3}(\mathbf{I} - 3\hat{\mathbf{r}}\hat{\mathbf{r}})\exp(\mathrm{i}k_o r), \tag{7}$$

$$\mathbf{g}(\mathbf{r})_{\mathrm{MF}} = \frac{\mathrm{i}}{k_o r^2}(\mathbf{I} - 3\hat{\mathbf{r}}\hat{\mathbf{r}})\exp(\mathrm{i}k_o r), \tag{8}$$

$$\mathbf{g}(\mathbf{r})_{\mathrm{FF}} = \frac{1}{r}(\mathbf{I} - \hat{\mathbf{r}}\hat{\mathbf{r}})\exp(\mathrm{i}k_o r). \tag{9}$$

The first contribution, $\mathbf{g}(\mathbf{r})_{\mathrm{SF}}$, is proportional to the delta function at $r = 0$, and this part is called the self-field. The next three terms have r dependences of r^{-3}, r^{-2} and r^{-1}, and these terms represent the near-, middle- and far-field dyadic Green's functions, respectively. Each part of the Green's function then determines the corresponding part of the electric field according to

$$\mathbf{E}(\mathbf{r})_{\alpha\mathrm{F}} = \frac{\mathrm{i}\omega\mu_o}{4\pi}\int \mathrm{d}^3\mathbf{r}'\mathbf{g}(\mathbf{r}-\mathbf{r}')_{\alpha\mathrm{F}}\cdot\mathbf{j}(\mathbf{r}'), \quad \alpha = \mathrm{S, N, M\ or\ F}. \tag{10}$$

For the self-field we then obtain

$$\mathbf{E}(\mathbf{r})_{\mathrm{SF}} = -\frac{\mathrm{i}}{3\varepsilon_o\omega}\mathbf{j}(\mathbf{r}), \tag{11}$$

indicating that this part is proportional to the current density at the same location.

3. Transverse and longitudinal components

A different way to split the electric field (or any other vector field) is into its transverse (t) and longitudinal (ℓ) components. Given $\mathbf{E}(\mathbf{r})$, these components are defined as

$$\mathbf{E}(\mathbf{r})^{(\mathrm{t})} = \frac{1}{4\pi}\nabla\times\left(\nabla\times\int \mathrm{d}^3\mathbf{r}'\mathbf{E}(\mathbf{r}')\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right), \tag{12}$$

$$\mathbf{E}(\mathbf{r})^{(\ell)} = -\frac{1}{4\pi}\nabla\left(\nabla\cdot\int \mathrm{d}^3\mathbf{r}'\mathbf{E}(\mathbf{r}')\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right), \tag{13}$$

and it follows immediately that these new fields have zero divergence and curl, respectively:

$$\nabla\cdot\mathbf{E}(\mathbf{r})^{(\mathrm{t})} = 0, \tag{14}$$

$$\nabla\times\mathbf{E}(\mathbf{r})^{(\ell)} = 0. \tag{15}$$

Just as for the Green's function, we can move the differential operators in (12) and (13) under the integral sign, and this yields an extra term reminiscent of the self-field part in (5). It then follows that we can represent the field components in the compact form:

$$\mathbf{E}(\mathbf{r})^{(\mathrm{t})} = \frac{2}{3}\mathbf{E}(\mathbf{r}) + \frac{1}{4\pi}\int \mathrm{d}^3\mathbf{r}'\boldsymbol{\eta}(\mathbf{r}-\mathbf{r}')\cdot\mathbf{E}(\mathbf{r}'), \tag{16}$$

$$\mathbf{E}(\mathbf{r})^{(\ell)} = \frac{1}{3}\mathbf{E}(\mathbf{r}) - \frac{1}{4\pi}\int \mathrm{d}^3\mathbf{r}'\boldsymbol{\eta}(\mathbf{r}-\mathbf{r}')\cdot\mathbf{E}(\mathbf{r}'), \tag{17}$$

and here the tensor $\boldsymbol{\eta}(\mathbf{r})$ is defined as

$$\boldsymbol{\eta}(\mathbf{r}) = \nabla\nabla\frac{1}{r}. \tag{18}$$

Explicitly,

$$\boldsymbol{\eta}(\mathbf{r}) = \frac{1}{r^3}(3\hat{\mathbf{r}}\hat{\mathbf{r}} - \mathbf{I}). \tag{19}$$

From expressions (16) and (17) it is evident that the sum of the transverse and longitudinal components equals the total field.

In the definitions above of the t and ℓ components of the electric field, these components are expressed in terms of the total field itself. When splitting the electric field, it is desirable to express the field components in terms of the source $\mathbf{j}(\mathbf{r})$ of the field, just like in (4) and (10) for the total field and the separate parts of the field, respectively. Therefore, we seek dyadic Green's functions $\mathbf{g}(\mathbf{r})^{(\beta)}$, with $\beta = \mathrm{t}$ or ℓ, such that

$$\mathbf{E}(\mathbf{r})^{(\beta)} = \frac{\mathrm{i}\omega\mu_{\mathrm{o}}}{4\pi}\int \mathrm{d}^3\mathbf{r}'\mathbf{g}(\mathbf{r}-\mathbf{r}')^{(\beta)}\cdot\mathbf{j}(\mathbf{r}'), \quad \beta = \mathrm{t} \text{ or } \ell. \tag{20}$$

These dyadic Green's functions will be evaluated explicitly in section 5.

4. Reciprocal space

An extremely useful tool for the study of the various fields is a transform from configuration ($\mathbf{r}$) space to reciprocal ($\mathbf{k}$) space. The transform of the electric field $\mathbf{E}(\mathbf{r})$ is defined as

$$\hat{\mathbf{E}}(\mathbf{k}) = \int \mathrm{d}^3\mathbf{r}\mathbf{E}(\mathbf{r})\exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{r}), \tag{21}$$

with inverse

$$\mathbf{E}(\mathbf{r}) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathbf{k}\hat{\mathbf{E}}(\mathbf{k})\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}), \tag{22}$$

and other fields transform similarly. Such Fourier transform pairs will be denoted as $\mathbf{E}(\mathbf{r}) \leftrightarrow \hat{\mathbf{E}}(\mathbf{k})$.

The transform of the scalar Green's function $g(\mathbf{r})$ will be indicated by $G(\mathbf{k})$, and is found to be

$$g(\mathbf{r}) = \frac{\exp(\mathrm{i}k_{\mathrm{o}}r)}{r} \leftrightarrow G(\mathbf{k}) = \frac{4\pi}{k^2 - k_{\mathrm{o}}^2 - \mathrm{i}\varepsilon}, \tag{23}$$

as can be verified by evaluating the transform integral as in (21). Here it is understood that we take $\varepsilon \downarrow 0$ whenever appropriate. This construction with ε is necessary for the inverse transform to reproduce $g(\mathbf{r})$.

For $k_{\mathrm{o}} = 0$, (23) reduces to

$$\frac{1}{4\pi r} \leftrightarrow \frac{1}{k^2 - \mathrm{i}\varepsilon}, \tag{24}$$

and with the convolution theorem we then find

$$\frac{1}{4\pi}\int \mathrm{d}^3\mathbf{r}'\mathbf{E}(\mathbf{r}')\frac{1}{|\mathbf{r}-\mathbf{r}'|} \leftrightarrow \frac{1}{k^2 - \mathrm{i}\varepsilon}\hat{\mathbf{E}}(\mathbf{k}). \tag{25}$$

For the transform of $\mathbf{E}(\mathbf{r})^{(\beta)}$ we shall write $\hat{\mathbf{E}}(\mathbf{k})^{(\beta)}$. Furthermore we have the symbolic relation $\nabla \leftrightarrow \mathrm{i}\mathbf{k}$, which then yields for the transforms of (12) and (13)

$$\hat{\mathbf{E}}(\mathbf{k})^{(\mathrm{t})} = -\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\hat{\mathbf{E}}(\mathbf{k})), \tag{26}$$

$$\hat{\mathbf{E}}(\mathbf{k})^{(\ell)} = \hat{\mathbf{k}}(\hat{\mathbf{k}}\cdot\hat{\mathbf{E}}(\mathbf{k})), \tag{27}$$

where $\hat{\mathbf{k}}$ represents the unit vector in the direction of $\mathbf{k}$, and where the limit $\varepsilon \downarrow 0$ has been taken. With a vector identity we then have

$$\hat{\mathbf{E}}(\mathbf{k})^{(\mathrm{t})} + \hat{\mathbf{E}}(\mathbf{k})^{(\ell)} = \hat{\mathbf{E}}(\mathbf{k}). \tag{28}$$

Similarly, the transform of (1) becomes

$$\hat{\mathbf{E}}(\mathbf{k}) = \frac{\mathrm{i}\omega\mu_\mathrm{o}}{4\pi} G(\mathbf{k}) \left[\mathbf{J}(\mathbf{k}) - \frac{1}{k_\mathrm{o}^2} \mathbf{k}(\mathbf{k} \cdot \mathbf{J}(\mathbf{k})) \right], \tag{29}$$

where $\mathbf{j}(\mathbf{r}) \leftrightarrow \mathbf{J}(\mathbf{k})$. Then we apply the operation on the right-hand side of (27) to (29) to obtain the longitudinal component:

$$\hat{\mathbf{E}}(\mathbf{k})^{(\ell)} = \frac{\mathrm{i}\omega\mu_\mathrm{o}}{4\pi} G(\mathbf{k}) \left(1 - \frac{k^2}{k_\mathrm{o}^2} \right) \hat{\mathbf{k}}(\hat{\mathbf{k}} \cdot \mathbf{J}(\mathbf{k})). \tag{30}$$

This can be simplified to

$$\hat{\mathbf{E}}(\mathbf{k})^{(\ell)} = -\frac{\mathrm{i}\omega\mu_\mathrm{o}}{k_\mathrm{o}^2} \hat{\mathbf{k}}(\hat{\mathbf{k}} \cdot \mathbf{J}(\mathbf{k})), \tag{31}$$

in view of (23). On the other hand, the transform of (20) is

$$\hat{\mathbf{E}}(\mathbf{k})^{(\beta)} = \frac{\mathrm{i}\omega\mu_\mathrm{o}}{4\pi} \mathbf{G}(\mathbf{k})^{(\beta)} \cdot \mathbf{J}(\mathbf{k}), \tag{32}$$

with $\mathbf{g}(\mathbf{r})^{(\beta)} \leftrightarrow \mathbf{G}(\mathbf{k})^{(\beta)}$. For $\beta = \ell$ this is the same as (31) provided we set

$$\mathbf{G}(\mathbf{k})^{(\ell)} = -\frac{4\pi}{k_\mathrm{o}^2} \hat{\mathbf{k}}\hat{\mathbf{k}}, \tag{33}$$

which is the longitudinal Green's function in $\mathbf{k}$ space. The transverse Green's function can be found along similar lines and the result is

$$\mathbf{G}(\mathbf{k})^{(\mathrm{t})} = G(\mathbf{k})(\mathbf{I} - \hat{\mathbf{k}}\hat{\mathbf{k}}). \tag{34}$$

On the other hand, when we write (29) in dyadic form as

$$\hat{\mathbf{E}}(\mathbf{k}) = \frac{\mathrm{i}\omega\mu_\mathrm{o}}{4\pi} \mathbf{G}(\mathbf{k}) \cdot \mathbf{J}(\mathbf{k}), \tag{35}$$

it follows that the dyadic Green's function for the total field is

$$\mathbf{G}(\mathbf{k}) = G(\mathbf{k}) \left(\mathbf{I} - \frac{k^2}{k_\mathrm{o}^2} \hat{\mathbf{k}}\hat{\mathbf{k}} \right). \tag{36}$$

From (33) and (34) it can then be verified by inspection that

$$\mathbf{G}(\mathbf{k})^{(\mathrm{t})} + \mathbf{G}(\mathbf{k})^{(\ell)} = \mathbf{G}(\mathbf{k}). \tag{37}$$

5. Transformation to configuration space

We now turn to the evaluation of the transverse and longitudinal Green's functions in $\mathbf{r}$ space, which generate the transverse and longitudinal components of the electric field from the current density, as shown in (20). The representation of the longitudinal Green's function in $\mathbf{k}$ space is given by (33), showing that it is equal to $\hat{\mathbf{k}}\hat{\mathbf{k}}$, apart from a constant. This dyadic $\hat{\mathbf{k}}\hat{\mathbf{k}}$ also appears on the right-hand side of (27), if we write this as $\hat{\mathbf{E}}(\mathbf{k})^{(\ell)} = (\hat{\mathbf{k}}\hat{\mathbf{k}}) \cdot \hat{\mathbf{E}}(\mathbf{k})$. The equivalent in $\mathbf{r}$ space is (17), which can be written as

$$\mathbf{E}(\mathbf{r})^{(\ell)} = \int \mathrm{d}^3\mathbf{r}' \boldsymbol{\delta}(\mathbf{r} - \mathbf{r}')^{(\ell)} \cdot \mathbf{E}(\mathbf{r}'), \tag{38}$$

so that

$$\boldsymbol{\delta}(\mathbf{r})^{(\ell)} = \frac{1}{3}\boldsymbol{\delta}(\mathbf{r}) - \frac{1}{4\pi}\boldsymbol{\eta}(\mathbf{r}). \tag{39}$$

This is the longitudinal delta function, which is the inverse transform of $\hat{\mathbf{k}}\hat{\mathbf{k}}$. Therefore, the longitudinal Green's function is

$$\mathbf{g}(\mathbf{r})^{(\ell)} = -\frac{4\pi}{k_o^2}\boldsymbol{\delta}(\mathbf{r})^{(\ell)}, \tag{40}$$

which is explicitly

$$\mathbf{g}(\mathbf{r})^{(\ell)} = -\frac{4\pi}{3k_o^2}\boldsymbol{\delta}(\mathbf{r}) + \frac{1}{k_o^2 r^3}(3\hat{\mathbf{r}}\hat{\mathbf{r}} - \mathbf{I}). \tag{41}$$

The transverse part can now be found immediately by noting that

$$\mathbf{g}(\mathbf{r})^{(\mathrm{t})} = \mathbf{g}(\mathbf{r}) - \mathbf{g}(\mathbf{r})^{(\ell)}, \tag{42}$$

where $\mathbf{g}(\mathbf{r})$ is given by (5), and the explicit representation is the sum of the four parts given in (6)–(9). We then obtain

$$\mathbf{g}(\mathbf{r})^{(\mathrm{t})} = (\mathbf{I} - \hat{\mathbf{r}}\hat{\mathbf{r}})\frac{\exp(\mathrm{i}k_o r)}{r} + (\mathbf{I} - 3\hat{\mathbf{r}}\hat{\mathbf{r}})\left[\frac{\mathrm{i}}{k_o r^2}\exp(\mathrm{i}k_o r) + \frac{1}{k_o^2 r^3}(1 - \exp(\mathrm{i}k_o r))\right]. \tag{43}$$

It is interesting to note that both the longitudinal and transverse components acquire an r^{-3} part. It should also be mentioned that $\mathbf{g}(\mathbf{r})^{(\mathrm{t})}$ can alternatively be obtained from its representation in $\mathbf{k}$ space, $\mathbf{G}(\mathbf{k})^{(\mathrm{t})} = G(\mathbf{k})(\mathbf{I} - \hat{\mathbf{k}}\hat{\mathbf{k}})$, (34), by applying directly the inverse integral transform, as in (22). We have verified that this leads to the same result.

6. The parts of the field in reciprocal space

We shall now split the four parts of the Green's function into their transverse and longitudinal components, leading to eight separate contributions. Just as for the total Green's function, this splitting is accomplished first in $\mathbf{k}$ space, and then a subsequent inverse transform yields the various components in $\mathbf{r}$ space.

To this end, we need the spatial transforms of the four Green's functions in (6)–(9). This can be done by direct integration, e.g. by evaluating the integrals

$$\mathbf{G}(\mathbf{k})_{\alpha\mathrm{F}} = \int \mathrm{d}^3\mathbf{r}\,\mathbf{g}(\mathbf{r})_{\alpha\mathrm{F}} \exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{r}), \quad \alpha = \mathrm{S, N, M\ or\ F}. \tag{44}$$

These integrals are rather cumbersome and have been studied in [12]. The result is

$$\mathbf{G}(\mathbf{k})_{\mathrm{SF}} = -\frac{4\pi}{3k_{\mathrm{o}}^2}\mathbf{I}, \tag{45}$$

$$\mathbf{G}(\mathbf{k})_{\mathrm{NF}} = \frac{4\pi}{k_{\mathrm{o}}^2}(\mathbf{I} - 3\hat{\mathbf{k}}\hat{\mathbf{k}})T(k_{\mathrm{o}}/k)_{\mathrm{NF}}, \tag{46}$$

$$\mathbf{G}(\mathbf{k})_{\mathrm{MF}} = \frac{4\pi}{k_{\mathrm{o}}^2}(\mathbf{I} - 3\hat{\mathbf{k}}\hat{\mathbf{k}})T(k_{\mathrm{o}}/k)_{\mathrm{MF}}, \tag{47}$$

$$\mathbf{G}(\mathbf{k})_{\mathrm{FF}} = (\mathbf{I} - \hat{\mathbf{k}}\hat{\mathbf{k}})G(\mathbf{k}) + \frac{4\pi}{k_{\mathrm{o}}^2}(\mathbf{I} - 3\hat{\mathbf{k}}\hat{\mathbf{k}})T(k_{\mathrm{o}}/k)_{\mathrm{FF}}. \tag{48}$$

The k dependence of these Green's functions enters through a set of auxiliary functions $T(q)_{\alpha\mathrm{F}}$, defined by the integral representations

$$T(q)_{\mathrm{NF}} = -\int_0^\infty \frac{\mathrm{d}t}{t^3}\left(3\cos t + \left(t - \frac{3}{t}\right)\sin t\right)\exp(\mathrm{i}qt), \tag{49}$$

$$T(q)_{\mathrm{MF}} = \mathrm{i}q\int_0^\infty \frac{\mathrm{d}t}{t^2}\left(3\cos t + \left(t - \frac{3}{t}\right)\sin t\right)\exp(\mathrm{i}qt), \tag{50}$$

$$T(q)_{\mathrm{FF}} = q^2\int_0^\infty \frac{\mathrm{d}t}{t}\left(\cos t - \frac{\sin t}{t}\right)\exp(\mathrm{i}qt), \tag{51}$$

where $q = k_{\mathrm{o}}/k$. These three universal functions are plotted in figures 1–3 as a function of $1/q$, which is the wave number k in units of k_{o}.

With these Green's functions, the parts of the field in $\mathbf{k}$ space are then

$$\hat{\mathbf{E}}(\mathbf{k})_{\alpha\mathrm{F}} = \frac{\mathrm{i}\omega\mu_{\mathrm{o}}}{4\pi}\mathbf{G}(\mathbf{k})_{\alpha\mathrm{F}}\cdot\mathbf{J}(\mathbf{k}). \tag{52}$$

It can be shown that the functions $T(q)_{\alpha\mathrm{F}}$ obey the sum rule

$$T(q)_{\mathrm{NF}} + T(q)_{\mathrm{MF}} + T(q)_{\mathrm{FF}} = \tfrac{1}{3}, \tag{53}$$

and with this identity we verify that the sum of the Green's functions is indeed the Green's function of the unsplit field:

$$\sum_\alpha \mathbf{G}(\mathbf{k})_{\alpha\mathrm{F}} = \mathbf{G}(\mathbf{k}). \tag{54}$$

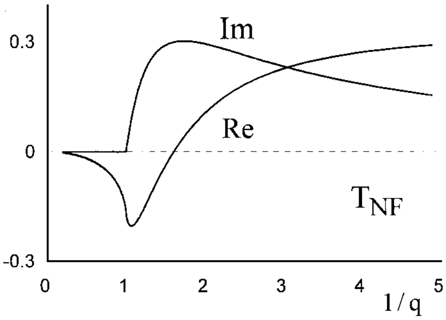

Figure 1. Graph of $T(q)_{\mathrm{NF}}$ plotted as a function of $1/q = k/k_{\mathrm{o}}$. For $k < k_{\mathrm{o}}$, the imaginary part of $T(q)_{\mathrm{NF}}$ is identically zero. The real part of $T(q)_{\mathrm{NF}}$ approaches the value 1/3 for $k \to \infty$.

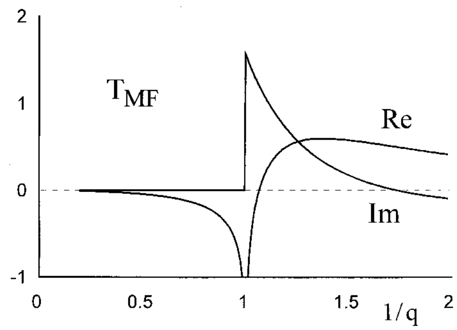

Figure 2. Graph of $T(q)_{\mathrm{MF}}$ plotted as a function of $1/q = k/k_{\mathrm{o}}$. At $k = k_{\mathrm{o}}$, the real part of $T(q)_{\mathrm{MF}}$ has a singularity and the imaginary part vanishes for $k < k_{\mathrm{o}}$.

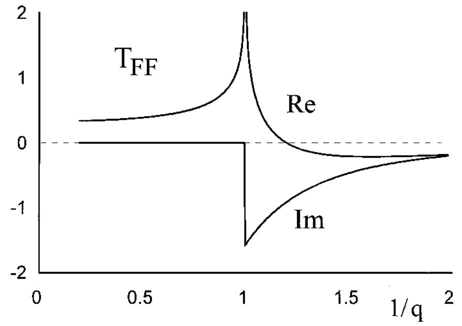

Figure 3. Graph of $T(q)_{\mathrm{FF}}$ plotted as a function of $1/q = k/k_{\mathrm{o}}$.

7. The components of the parts of the field in k space

We now seek the Green's functions which filter out the transverse and longitudinal components of the parts, e.g. when operating on the current density, the result is

$$\hat{\mathbf{E}}(\mathbf{k})^{(\beta)}_{\alpha\mathrm{F}} = \frac{\mathrm{i}\omega\mu_{\mathrm{o}}}{4\pi}\mathbf{G}(\mathbf{k})^{(\beta)}_{\alpha\mathrm{F}} \cdot \mathbf{J}(\mathbf{k}). \tag{55}$$

In $\mathbf{k}$ space this splitting is particularly simple since the dyadic operator $\hat{\mathbf{k}}\hat{\mathbf{k}}$ filters out the longitudinal part, and therefore the operator $\mathbf{I} - \hat{\mathbf{k}}\hat{\mathbf{k}}$ yields the transverse part. It then follows immediately that the Green's functions from (45)–(48) separate as

$$\mathbf{G}(\mathbf{k})^{(\mathrm{t})}_{\mathrm{SF}} = -\frac{4\pi}{3k_{\mathrm{o}}^2}(\mathbf{I} - \hat{\mathbf{k}}\hat{\mathbf{k}}), \tag{56}$$

$$\mathbf{G}(\mathbf{k})^{(\ell)}_{\mathrm{SF}} = -\frac{4\pi}{3k_{\mathrm{o}}^2}\hat{\mathbf{k}}\hat{\mathbf{k}}, \tag{57}$$

$$\mathbf{G}(\mathbf{k})^{(\mathrm{t})}_{\mathrm{NF}} = \frac{4\pi}{k_{\mathrm{o}}^2}(\mathbf{I} - \hat{\mathbf{k}}\hat{\mathbf{k}})\,T(k_{\mathrm{o}}/k)_{\mathrm{NF}}, \tag{58}$$

$$\mathbf{G}(\mathbf{k})^{(\ell)}_{\mathrm{NF}} = -\frac{8\pi}{k_{\mathrm{o}}^2}\hat{\mathbf{k}}\hat{\mathbf{k}}\,T(k_{\mathrm{o}}/k)_{\mathrm{NF}}, \tag{59}$$

$$\mathbf{G}(\mathbf{k})^{(\mathrm{t})}_{\mathrm{MF}} = \frac{4\pi}{k_{\mathrm{o}}^2}(\mathbf{I} - \hat{\mathbf{k}}\hat{\mathbf{k}})T(k_{\mathrm{o}}/k)_{\mathrm{MF}}, \tag{60}$$

$$\mathbf{G}(\mathbf{k})^{(\ell)}_{\mathrm{MF}} = -\frac{8\pi}{k_{\mathrm{o}}^2}\hat{\mathbf{k}}\hat{\mathbf{k}}T(k_{\mathrm{o}}/k)_{\mathrm{MF}}, \tag{61}$$

$$\mathbf{G}(\mathbf{k})^{(\mathrm{t})}_{\mathrm{FF}} = (\mathbf{I} - \hat{\mathbf{k}}\hat{\mathbf{k}})\left(G(\mathbf{k}) + \frac{4\pi}{k_{\mathrm{o}}^2}T(k_{\mathrm{o}}/k)_{\mathrm{FF}}\right), \tag{62}$$

$$\mathbf{G}(\mathbf{k})^{(\ell)}_{\mathrm{FF}} = -\frac{8\pi}{k_{\mathrm{o}}^2}\hat{\mathbf{k}}\hat{\mathbf{k}}T(k_{\mathrm{o}}/k)_{\mathrm{FF}}. \tag{63}$$

It is interesting to see that the first term of the transverse part of the far-field, $(\mathbf{I} - \hat{\mathbf{k}}\hat{\mathbf{k}})G(\mathbf{k})$, is just the transverse Green's function of the total field, as given in (34). Also, with the sum rule (53), we notice that the Green's functions for the transverse (longitudinal) parts add up to the transverse (longitudinal) Green's function for the total field.

8. The components of the parts of the field in r space

The final step is to obtain the various Green's functions in $\mathbf{r}$ space. We let

$$\mathbf{g}(\mathbf{r})^{(\beta)}_{\alpha\mathrm{F}} \leftrightarrow \mathbf{G}(\mathbf{k})^{(\beta)}_{\alpha\mathrm{F}}, \tag{64}$$

which generates the β component of the α part of the field from the current density according to

$$\mathbf{E}(\mathbf{r})^{(\beta)}_{\alpha\mathrm{F}} = \frac{\mathrm{i}\omega\mu_{\mathrm{o}}}{4\pi}\int \mathrm{d}^3\mathbf{r}'\mathbf{g}(\mathbf{r}-\mathbf{r}')^{(\beta)}_{\alpha\mathrm{F}}\cdot\mathbf{j}(\mathbf{r}'). \tag{65}$$

Let us first consider the self-field. As shown in section 5, we have $\boldsymbol{\delta}(\mathbf{r})^{(\ell)} \leftrightarrow \hat{\mathbf{k}}\hat{\mathbf{k}}$, and similarly we have $\boldsymbol{\delta}(\mathbf{r})^{(\mathrm{t})} \leftrightarrow \mathbf{I} - \hat{\mathbf{k}}\hat{\mathbf{k}}$. Therefore, the Green's functions for the components of the self-field are

$$\mathbf{g}(\mathbf{r})^{(\mathrm{t})}_{\mathrm{SF}} = -\frac{4\pi}{3k_{\mathrm{o}}^2}\boldsymbol{\delta}(\mathbf{r})^{(\mathrm{t})}, \tag{66}$$

$$\mathbf{g}(\mathbf{r})^{(\ell)}_{\mathrm{SF}} = -\frac{4\pi}{3k_{\mathrm{o}}^2}\boldsymbol{\delta}(\mathbf{r})^{(\ell)}, \tag{67}$$

or more explicitly

$$\mathbf{g}(\mathbf{r})^{(\mathrm{t})}_{\mathrm{SF}} = -\frac{8\pi}{9k_{\mathrm{o}}^2}\boldsymbol{\delta}(\mathbf{r}) - \frac{1}{3k_{\mathrm{o}}^2r^3}(3\hat{\mathbf{r}}\hat{\mathbf{r}} - \mathbf{I}), \tag{68}$$

$$\mathbf{g}(\mathbf{r})^{(\ell)}_{\mathrm{SF}} = -\frac{4\pi}{9k_{\mathrm{o}}^2}\boldsymbol{\delta}(\mathbf{r}) + \frac{1}{3k_{\mathrm{o}}^2r^3}(3\hat{\mathbf{r}}\hat{\mathbf{r}} - \mathbf{I}). \tag{69}$$

Comparison with the Green's function of the longitudinal component of the total field, equation (40), gives

$$\mathbf{g}(\mathbf{r})^{(\ell)}_{\mathrm{SF}} = \frac{1}{3}\mathbf{g}(\mathbf{r})^{(\ell)}, \tag{70}$$

showing that 1/3 of the total longitudinal field ends up in the self-field. On the other hand, (41) combined with (6) yields

$$\mathbf{g}(\mathbf{r})^{(\ell)} = \mathbf{g}(\mathbf{r})_{\mathrm{SF}} + \frac{1}{k_{\mathrm{o}}^2 r^3}(3\hat{\mathbf{r}}\hat{\mathbf{r}} - \mathbf{I}), \tag{71}$$

where $\mathbf{g}(\mathbf{r})_{\mathrm{SF}}$ is the delta function contribution to the total field. So, it seems that the delta function part of the field only contributes to the longitudinal component. However, the longitudinal component of this self-field only accounts for 1/3 of the total delta function part of the longitudinal field. This seemingly contradictory result can be explained by noting that 'the component of a part' is not necessarily the same as 'the part of a component', e.g. these operations do not commute. For instance, the transverse component of the self-field has an r^{-3} part, which is of the near-field type. We shall see below that the other 2/3 of the delta function in the total longitudinal component comes from the near-field. But then, since the near-field itself does not have a delta function, this implies that the transverse component of the near-field must have a delta function contribution, even though the total transverse field does not.

The remaining six Green's functions will be evaluated by direct integration:

$$\mathbf{g}(\mathbf{r})_{\alpha\mathrm{F}}^{(\beta)} = \frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathbf{k}\mathbf{G}(\mathbf{k})_{\alpha\mathrm{F}}^{(\beta)} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}), \tag{72}$$

except for the part $(\mathbf{I} - \hat{\mathbf{k}}\hat{\mathbf{k}})G(\mathbf{k})$ in the transverse component of the far-field, since we already know that its inverse is $\mathbf{g}(\mathbf{r})^{(\mathrm{t})}$ from (43). Then each remaining term to be inverted has a function $T(k_{\mathrm{o}}/k)_{\alpha\mathrm{F}}$ in it, for which we shall use the integral representations from section 6. For $\alpha = \mathrm{M}$ or F these functions go to zero sufficiently fast for $k \to \infty$ for the integral in (72) to exist, but for the near-field we have $T(k_{\mathrm{o}}/k)_{\mathrm{NF}} \to 1/3$ for $k \to \infty$. This leads again to delta function contributions, which will be split off first. We write $T_{\mathrm{NF}} = 1/3 + (T_{\mathrm{NF}} - 1/3)$ and consider the effect of the 1/3 first. From (58) and (59) we see that this 1/3 leads to

$$\mathbf{g}(\mathbf{r})_{\mathrm{NF}}^{(\mathrm{t})} = \frac{4\pi}{3k_{\mathrm{o}}^2}\boldsymbol{\delta}(\mathbf{r})^{(\mathrm{t})} + \{\ldots\}, \tag{73}$$

$$\mathbf{g}(\mathbf{r})_{\mathrm{NF}}^{(\ell)} = -\frac{8\pi}{3k_{\mathrm{o}}^2}\boldsymbol{\delta}(\mathbf{r})^{(\ell)} + \{\ldots\}, \tag{74}$$

and here $\{\ldots\}$ stands for the inverse with T_{NF} replaced by $T_{\mathrm{NF}} - 1/3$. We notice that this '1/3 contribution' to $\mathbf{g}(\mathbf{r})_{\mathrm{NF}}^{(\mathrm{t})}$ is just the opposite of $\mathbf{g}(\mathbf{r})_{\mathrm{SF}}^{(\mathrm{t})}$, showing that the total transverse field does not have a delta function indeed. On the other hand, for the longitudinal component we get from the self-field and near-field

$$\begin{aligned}\mathbf{g}(\mathbf{r})_{\mathrm{SF}}^{(\ell)} + \mathbf{g}(\mathbf{r})_{\mathrm{NF}}^{(\ell)} &= -\frac{4\pi}{3k_{\mathrm{o}}^2}\boldsymbol{\delta}(\mathbf{r})^{(\ell)} - \frac{8\pi}{3k_{\mathrm{o}}^2}\boldsymbol{\delta}(\mathbf{r})^{(\ell)} + \{\ldots\} \\ &= -\frac{4\pi}{k_{\mathrm{o}}^2}\boldsymbol{\delta}(\mathbf{r})^{(\ell)} + \{\ldots\},\end{aligned} \tag{75}$$

and here the first term on the right-hand side is exactly $\mathbf{g}(\mathbf{r})^{(\ell)}$ from (40). Therefore, all delta functions have been accounted for at this point. It also shows that the sum of the longitudinal components of the middle- and far-field must equal the opposite of $\{\ldots\}$ in (75). Obviously, this is equivalent to the sum rule (53).

For the inverse transforms of the terms with a $T_{\alpha\mathrm{F}}$ function we proceed by direct integration. First we consider the longitudinal components, which are of the form $\hat{\mathbf{k}}\hat{\mathbf{k}}T(k_\mathrm{o}/k)_{\alpha\mathrm{F}}$, apart from an overall constant. For the middle- and far-field we use the integral representations (50) and (51) for T_MF and T_FF, respectively. For the near-field we replace T_NF by $T_\mathrm{NF}-1/3$. From (49) we can derive the following representation for $T_\mathrm{NF}-1/3$:

$$T(q)_\mathrm{NF}-\frac{1}{3}=-\mathrm{i}q\int_0^\infty\frac{\mathrm{d}t}{t^3}(t\cos t-\sin t)\exp(\mathrm{i}qt). \tag{76}$$

In the Appendix we show in some detail how these inverse transforms are performed. The results for the longitudinal components are:

$$\mathbf{g}(\mathbf{r})^{(\ell)}_\mathrm{NF}=-\frac{8\pi}{9k_\mathrm{o}^2}\boldsymbol{\delta}(\mathbf{r})+\frac{2}{k_\mathrm{o}^2r^3}(\hat{\mathbf{r}}\hat{\mathbf{r}}\exp(\mathrm{i}k_\mathrm{o}r)-\mathbf{I}\zeta_4(k_\mathrm{o}r)), \tag{77}$$

$$\mathbf{g}(\mathbf{r})^{(\ell)}_\mathrm{MF}=-\frac{2\mathrm{i}}{k_\mathrm{o}r^2}(\hat{\mathbf{r}}\hat{\mathbf{r}}\exp(\mathrm{i}k_\mathrm{o}r)-\mathbf{I}\zeta_3(k_\mathrm{o}r)), \tag{78}$$

$$\mathbf{g}(\mathbf{r})^{(\ell)}_\mathrm{FF}=-\frac{2}{3k_\mathrm{o}r^2}(\mathbf{I}-3\hat{\mathbf{r}}\hat{\mathbf{r}})\left[\mathrm{i}\exp(\mathrm{i}k_\mathrm{o}r)-\frac{1}{k_\mathrm{o}r}(\exp(\mathrm{i}k_\mathrm{o}r)-1)\right]+\frac{2}{3r}\mathbf{I}\zeta_2(k_\mathrm{o}r). \tag{79}$$

Here we have introduced the auxiliary functions

$$\zeta_n(x)=\int_1^\infty \mathrm{d}p\,\frac{\exp(\mathrm{i}px)}{p^n},\quad x>0,\quad n=1,2,\ldots, \tag{80}$$

needed for $n=2$, 3 and 4.

The functions $\zeta_n(x)$ satisfy the recurrence relation

$$\zeta_{n+1}(x)=\frac{1}{n}(\exp(\mathrm{i}x)+\mathrm{i}x\zeta_n(x)), \tag{81}$$

and therefore they can successively be obtained from $\zeta_1(x)$. In terms of the sine and cosine integrals $\mathrm{si}(x)$ and $\mathrm{ci}(x)$ this function is [13]

$$\zeta_1(x)=-\mathrm{ci}(x)-\mathrm{i}\,\mathrm{si}(x). \tag{82}$$

By repeated application of (81), going from a given n to higher ones, we find the asymptotic series

$$\zeta_n(x)=-\frac{\exp(\mathrm{i}x)}{\mathrm{i}x}\left[1+\frac{n}{\mathrm{i}x}+\frac{n(n+1)}{(\mathrm{i}x)^2}+\cdots\right],\quad x \text{ large}. \tag{83}$$

Therefore, the leading term of $\zeta_n(k_\mathrm{o}r)$ in the region of large r is $\mathrm{i}\exp(\mathrm{i}k_\mathrm{o}r)/k_\mathrm{o}r$, independent of n. From (77) we then notice that the longitudinal component of the near-field has an r^{-4} contribution, and the Green's function of the longitudinal component of the middle-field, given by (78), has an r^{-3} part. Interesting about the longitudinal part of the far-field is that it has no $1/r$ component. For the behaviour at small x, or r, we use the familiar results for $\mathrm{ci}(x)$ and $\mathrm{si}(x)$, which yields

$$\zeta_1(x)=-\gamma-\ln x+\frac{\mathrm{i}\pi}{2}+\mathcal{O}(x),\quad x\downarrow 0, \tag{84}$$

with γ Euler's constant. Then with (81) we find

$$\zeta_n(0)=\frac{1}{n-1},\quad n=2,3,\ldots. \tag{85}$$

This shows that $\zeta_n(x)$ for $n = 2$, 3 and 4 remains finite for $x \downarrow 0$. The functions $\zeta_2(x)$, $\zeta_3(x)$ and $\zeta_4(x)$ are illustrated in figures 4 and 5.

With (81), both the $\zeta_3(k_o r)$ in (78) and the $\zeta_4(k_o r)$ in (77) can be expressed in terms of $\zeta_2(k_o r)$. When we then add (77)–(79), and add the longitudinal component of the self-field from (69), we verify that

$$\sum_\alpha \mathbf{g}(\mathbf{r})^{(\ell)}_{\alpha\mathrm{F}} = \mathbf{g}(\mathbf{r})^{(\ell)}. \tag{86}$$

The Green's functions for the transverse components of the parts of the field now follow from

$$\mathbf{g}(\mathbf{r})^{(\mathrm{t})}_{\alpha\mathrm{F}} = \mathbf{g}(\mathbf{r})_{\alpha\mathrm{F}} - \mathbf{g}(\mathbf{r})^{(\ell)}_{\alpha\mathrm{F}}, \tag{87}$$

and the result is

$$\mathbf{g}(\mathbf{r})^{(\mathrm{t})}_{\mathrm{NF}} = \frac{8\pi}{9k_o^2}\boldsymbol{\delta}(\mathbf{r}) - \frac{1}{k_o^2 r^3}(\mathbf{I} - \hat{\mathbf{r}}\hat{\mathbf{r}})\exp(\mathrm{i}k_o r) + \frac{2}{k_o^2 r^3}\mathbf{I}\zeta_4(k_o r), \tag{88}$$

$$\mathbf{g}(\mathbf{r})^{(\mathrm{t})}_{\mathrm{MF}} = \frac{\mathrm{i}}{k_o r^2}(\mathbf{I} - \hat{\mathbf{r}}\hat{\mathbf{r}})\exp(\mathrm{i}k_o r) - \frac{2\mathrm{i}}{k_o r^2}\mathbf{I}\zeta_3(k_o r), \tag{89}$$

$$\begin{aligned}\mathbf{g}(\mathbf{r})^{(\mathrm{t})}_{\mathrm{FF}} &= (\mathbf{I} - \hat{\mathbf{r}}\hat{\mathbf{r}})\frac{\exp(\mathrm{i}k_o r)}{r} - \frac{2}{3r}\mathbf{I}\zeta_2(k_o r)\\ &\quad + \frac{2}{3k_o r^2}(\mathbf{I} - 3\hat{\mathbf{r}}\hat{\mathbf{r}})\left[\mathrm{i}\exp(\mathrm{i}k_o r) - \frac{1}{k_o r}(\exp(\mathrm{i}k_o r) - 1)\right].\end{aligned} \tag{90}$$

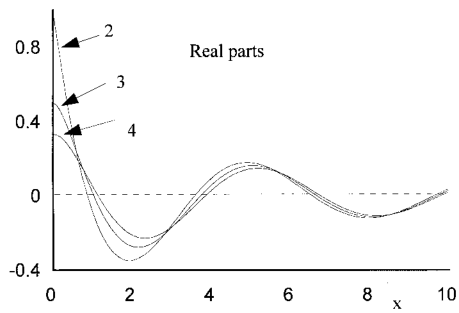

Figure 4. Plot of the real parts of the functions $\zeta_2(x)$, $\zeta_3(x)$ and $\zeta_4(x)$.

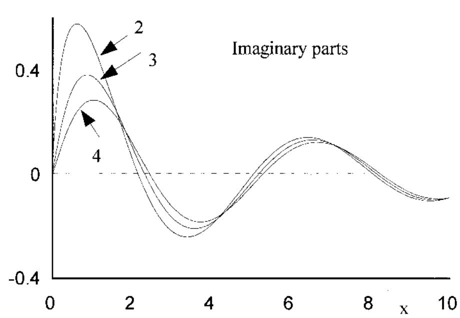

Figure 5. Plot of the imaginary parts of the functions $\zeta_2(x)$, $\zeta_3(x)$ and $\zeta_4(x)$. These imaginary parts are zero for $x = 0$.

9. Electric dipole

As an example, let us consider an electric dipole located in $r = 0$. The dipole moment is $\boldsymbol{\mu}(\omega)$ and the corresponding current density is

$$\mathbf{j}(\mathbf{r}) = -\mathrm{i}\omega\boldsymbol{\mu}\delta(\mathbf{r}). \tag{91}$$

With (65) we then find for the components of the parts of the field

$$\mathbf{E}(\mathbf{r})^{(\beta)}_{\alpha\mathrm{F}} = \frac{k_\mathrm{o}^2}{4\pi\varepsilon_\mathrm{o}}\mathbf{g}(\mathbf{r})^{(\beta)}_{\alpha\mathrm{F}} \cdot \boldsymbol{\mu}. \tag{92}$$

For instance, with (78) and (89), respectively, the longitudinal and transverse components of the middle-field are found to be

$$\mathbf{E}(\mathbf{r})^{(\ell)}_{\mathrm{MF}} = \frac{-\mathrm{i}k_\mathrm{o}}{2\pi\varepsilon_\mathrm{o}r^2}[\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \boldsymbol{\mu}) \exp(\mathrm{i}k_\mathrm{o}r) - \zeta_3(k_\mathrm{o}r)\boldsymbol{\mu}], \tag{93}$$

$$\mathbf{E}(\mathbf{r})^{(\mathrm{t})}_{\mathrm{MF}} = \frac{\mathrm{i}k_\mathrm{o}}{4\pi\varepsilon_\mathrm{o}r^2}\{[\boldsymbol{\mu} - \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \boldsymbol{\mu})] \exp(\mathrm{i}k_\mathrm{o}r) - 2\zeta_3(k_\mathrm{o}r)\boldsymbol{\mu}\}, \tag{94}$$

and similar expressions can be written down for the other three parts of the dipole field. With these explicit results we can now verify by differentiation that these fields are indeed longitudinal and transverse, e.g.

$$\nabla \times \mathbf{E}(\mathbf{r})^{(\ell)}_{\mathrm{MF}} = 0, \tag{95}$$

$$\nabla \cdot \mathbf{E}(\mathbf{r})^{(\mathrm{t})}_{\mathrm{MF}} = 0. \tag{96}$$

For this we need the derivatives of the $\zeta_n(x)$ functions. From the definition (80) it follows that

$$\zeta_n(x)' = \mathrm{i}\zeta_{n-1}(x), \quad n = 2, 3, \ldots, \tag{97}$$

and with the recursion relation (81) this becomes

$$\zeta_n(x)' = \frac{1}{x}[(n-1)\zeta_n(x) - \exp(\mathrm{i}x)], \quad n = 2, 3, \ldots, \tag{98}$$

expressing the derivative in terms of $\zeta_n(x)$ itself.

10. Conclusions

We have studied the dyadic Green's function which determines the electric field from a given current density. The electric field naturally separates into four parts: the self-, near-, middle- and far-field. This separation can be contributed to four corresponding parts of the dyadic Green's function. On the other hand, the electric field can be considered as a sum of a longitudinal and a transverse component, and this separation can also be accounted for by a corresponding splitting of the dyadic Green's function. But then, each of the four parts of the field, or the corresponding Green's function, has its own distinct contribution to both the longitudinal and to the transverse field. We have evaluated the longitudinal and transverse components of each part of the field, and this was done both in configuration space and in reciprocal space. In $\mathbf{k}$ space, this splitting involves a set of three auxiliary functions T_{NF}, T_{MF} and T_{FF}, which depend only on the magnitude of vector $\mathbf{k}$. In $\mathbf{r}$ space, the representation of the near-, middle- and far-field involves the auxiliary functions $\zeta_4(k_\mathrm{o}r)$, $\zeta_3(k_\mathrm{o}r)$ and $\zeta_2(k_\mathrm{o}r)$, respectively, and these depend only on the magnitude of vector $\mathbf{r}$. A particularly interesting result is that the Green's functions of the longitudinal and transverse components of the near-field acquire a delta function contribution, whereas the Green's function for the total near-field has no delta function part. Vice versa, the Green's function for the total self-field is proportional to a delta function, but the Green's functions for the longitudinal and transverse components have a continuous part.

Appendix

The Green's function of the longitudinal component of the far-field in $\mathbf{r}$ space is the inverse of (63), and with (22) this is

$$\mathbf{g}(\mathbf{r})^{(\ell)}_{\mathrm{FF}} = -\frac{1}{\pi^2 k_o^2}\int \mathrm{d}^3\mathbf{k}(\hat{\mathbf{k}}\hat{\mathbf{k}})T(k_o/k)_{\mathrm{FF}} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}). \tag{A 1}$$

In order to evaluate this integral we adopt spherical coordinates (k,θ,ϕ) in $\mathbf{k}$ space, and we take the polar axis along the vector $\mathbf{r}$. The volume element is $d^3\mathbf{k} = k^2 \sin\theta\,\mathrm{d}k\,\mathrm{d}\theta\,\mathrm{d}\phi$, and we have $\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}) = \exp(\mathrm{i}kr\cos\theta)$. The only ϕ dependence appears through $\hat{\mathbf{k}}\hat{\mathbf{k}}$ and this leads to

$$\int_0^{2\pi}\mathrm{d}\phi\hat{\mathbf{k}}\hat{\mathbf{k}} = (\mathbf{I}-\hat{\mathbf{r}}\hat{\mathbf{r}})\pi\sin^2\theta + \hat{\mathbf{r}}\hat{\mathbf{r}}2\pi\cos^2\theta. \tag{A 2}$$

Then the θ dependence comes in as in (A 2), through the volume element, and through $\exp(\mathrm{i}kr\cos\theta)$. Integration over θ yields

$$\mathbf{g}(\mathbf{r})^{(\ell)}_{\mathrm{FF}} = -\frac{4}{\pi k_o^2}\int_0^\infty \mathrm{d}k k^2 T(k_o/k)_{\mathrm{FF}} \times\left[\hat{\mathbf{r}}\hat{\mathbf{r}}\frac{\sin(kr)}{kr} + (3\hat{\mathbf{r}}\hat{\mathbf{r}}-\mathbf{I})\left(\frac{\cos(kr)}{(kr)^2}-\frac{\sin(kr)}{(kr)^3}\right)\right]. \tag{A 3}$$

Next we make the change of variables $u = kr$. This gives for the argument of the T function $k_o/k = k_o r/u$. Then we substitute the integral representation (51) for $T(k_o r/u)$, and subsequently we change the integration variable from t to $v = k_o rt/u$. This gives

$$\mathbf{g}(\mathbf{r})^{(\ell)}_{\mathrm{FF}} = -\frac{4}{\pi r}\int_0^\infty \mathrm{d}u\int_0^\infty \mathrm{d}v\frac{\exp(\mathrm{i}v)}{v}\left(\cos\left(\frac{uv}{k_o r}\right)-\frac{k_o r}{uv}\sin\left(\frac{uv}{k_o r}\right)\right) \times\left[\hat{\mathbf{r}}\hat{\mathbf{r}}\frac{\sin u}{u} + (3\hat{\mathbf{r}}\hat{\mathbf{r}}-\mathbf{I})\left(\frac{\cos u}{u^2}-\frac{\sin u}{u^3}\right)\right]. \tag{A 4}$$

Here, we integrate over u first. Care should be exercised about the lower limit, $u = 0$. When integrating by parts several times, the integral can be reduced to a combination of tabulated integrals. However, various integrated parts do not exist in the lower limit. Therefore, first we keep the lower limit finite, say $u = \delta$, and then in the end we take $\delta \downarrow 0$. This gives for the integral over u:

$$\int_0^\infty \mathrm{d}u\langle\ldots\rangle\langle\ldots\rangle = -\mathbf{I}\frac{\pi}{6}\begin{cases} p^2 \\ 1 \\ 1/p \end{cases} + \hat{\mathbf{r}}\hat{\mathbf{r}}\frac{\pi}{4}\begin{cases} 2p^2, & 0\le p<1, \\ 1, & p=1, \\ 0, & p>1, \end{cases} \tag{A 5}$$

where we have set $p = v/k_o r$. When we substitute (A 5) into (A 4), and change the integration variable to p we find

$$\mathbf{g}(\mathbf{r})^{(\ell)}_{\mathrm{FF}} = \frac{2}{3r}\left[(\mathbf{I} - 3\hat{\mathbf{r}}\hat{\mathbf{r}})\int_0^1 \mathrm{d}pp\, \exp(\mathrm{i}pk_o r) + \mathbf{I}\int_1^\infty \mathrm{d}p \frac{\exp(\mathrm{i}pk_o r)}{p^2}\right]. \tag{A 6}$$

The second integral is $\zeta_2(k_o r)$, and the first one can be evaluated easily. The result is (79).

For the Green's function of the longitudinal component of the middle-field we follow the same procedure. The equivalent of (A 4) becomes

$$\mathbf{g}(\mathbf{r})^{(\ell)}_{\mathrm{MF}} = -\frac{4\mathrm{i}}{\pi r}\int_0^\infty \mathrm{d}u \int_0^\infty \mathrm{d}v \frac{\exp(\mathrm{i}v)}{v^2}\left[3\cos\left(\frac{uv}{k_o r}\right) + \left(\frac{uv}{k_o r} - 3\frac{k_o r}{uv}\right)\sin\left(\frac{uv}{k_o r}\right)\right]$$
$$\times\left[\hat{\mathbf{r}}\hat{\mathbf{r}}\frac{\sin u}{u} + (3\hat{\mathbf{r}}\hat{\mathbf{r}} - \mathbf{I})\left(\frac{\cos u}{u^2} - \frac{\sin u}{u^3}\right)\right]. \tag{A 7}$$

Here we encounter the problem that strictly speaking the integral over u does not exist in the upper limit, for one of the terms. In that term we keep the upper limit $u = k_{\max} r$ finite for the time being. Then the integral over u becomes

$$\int_0^\infty \mathrm{d}u\langle\ldots\rangle\langle\ldots\rangle = \hat{\mathbf{r}}\hat{\mathbf{r}}p\int_0^{k_{\max}r} \mathrm{d}u \sin(pu)\sin u - \mathbf{I}\frac{\pi}{4}\begin{cases} 0, & 0 \le p < 1,\\ 1, & p = 1,\\ 2/p, & p > 1.\end{cases} \tag{A 8}$$

Substitution into (A 7) then gives, after changing the integration variable to p,

$$\mathbf{g}(\mathbf{r})^{(\ell)}_{\mathrm{MF}} = \frac{2\mathrm{i}}{k_o r^2}\mathbf{I}\zeta_3(k_o r) - \frac{4\mathrm{i}}{\pi k_o r^2}\hat{\mathbf{r}}\hat{\mathbf{r}}\int_0^{k_{\max}r}\mathrm{d}u \sin u \int_0^\infty \mathrm{d}p \frac{\exp(\mathrm{i}pk_o r)}{p}\sin(pu). \tag{A 9}$$

Here we integrate over p first, with the result

$$\int_0^\infty \mathrm{d}p\frac{\exp(\mathrm{i}pk_o r)}{p}\sin(pu) = \frac{\mathrm{i}}{4}\ln\left[\left(\frac{u + k_o r}{u - k_o r}\right)^2\right] + \begin{cases}\pi/2, & u > k_o r,\\ \pi/4, & u = k_o r,\\ 0, & u < k_o r.\end{cases} \tag{A 10}$$

Now we encounter the problem that the right-hand side has a singularity at $u = k_o r$, and when substituted into (A 9) we have to evaluate the integral over u as a principal value integral around $u = k_o r$. Since the singularity is logarithmic, this integral exists. The remaining integral over u with the logarithm in it exists for $k_{\max} \to \infty$. We split the integral in an integral over $[0, k_o r - \varepsilon]$ and one over $[k_o r + \varepsilon, \infty]$, integrate by parts, and then take the limit $\varepsilon \downarrow 0$. This gives

$$\int_0^\infty \mathrm{d}u(\sin u)\ln\left[\left(\frac{u + k_o r}{u - k_o r}\right)^2\right] = 4k_o r\int_0^\infty \mathrm{d}u\frac{\cos u}{k_o^2 r^2 - u^2}, \tag{A 11}$$

which is still a principal value integral. In order to evaluate this integral, we extend the range to $u = -\infty$, close the contour with a semicircle in the upper half of the complex u plane, avoiding the poles at $u = k_o r$ and $u = -k_o r$ with small semicircles above the real axis. Then the contour integral is zero, so the integral itself is the sum of the integrals over the small semicircles. This yields

$$\int_0^\infty \mathrm{d}u(\sin u)\ln\left[\left(\frac{u + k_o r}{u - k_o r}\right)^2\right] = 2\pi\sin(k_o r). \tag{A 12}$$

Putting everything together then gives for the Green's function

$$\mathbf{g}(\mathbf{r})^{(\ell)}_{\mathrm{MF}} = \frac{2\mathrm{i}}{k_o r^2}[\mathbf{I}\zeta_3(k_o r) - \hat{\mathbf{r}}\hat{\mathbf{r}}(\exp(\mathrm{i}k_o r) - \cos(k_{\max} r))]. \qquad \text{(A 13)}$$

Here we have taken $k_{\max} \to \infty$ wherever possible. The term with $\cos(k_{\max} r)$, however, does not formally exist. However, for $k_{\max}$ large, this term varies rapidly as a function of r. Since every Green's function is eventually integrated over, this term will average to zero. Therefore we can leave it out.

The Green's function for the longitudinal component of the near-field can be evaluated along the same lines, with no additional complications.

References

[1] Pohl, D. W., and Courjon, D. (eds), 1993, *Proceedings of the NATO Advanced Research Workshop on Near Field Optics, Series E, Applied Sciences*, Vol. 242 (Dordrecht: Kluwer).

[2] Paesler, M. A., and Moyer, P. J., 1996, *Near Field Optics, Theory, Instrumentation, and Applications* (New York: Wiley).

[3] Ohtsu, M. (ed.), 1998, *Near-Field Nano/Atom Optics and Technology* (Berlin: Springer).

[4] Grattan, K. T. V., and Meggitt, B. T. (eds), 2000, *Optical Fiber Sensor Technology, Fundamentals* (The Netherlands: Kluwer).

[5] Keller, O., 1996, *Phys. Rep.*, **268,** 85.

[6] Keller, O., 1999, *Computational Studies of New Materials* (Singapore: World Scientific), pp. 375–439.

[7] Keller, O., 1999, *J. opt. Soc. Am.* B, **16,** 835.

[8] Jackson, J. D., 1999, *Classical Electrodynamics*, 3rd edition (New York: Wiley), p. 149.

[9] Cohen-Tannoudji, C., Dupont-Roc, J., and Grynberg, G., 1989, *Photons & Atoms* (New York: Wiley).

[10] van Kranendonk, J., and Sipe, J. E., 1977, *Progress in Optics*, Vol. XV, edited by E. Wolf (Amsterdam: North-Holland), p. 246, ff.

[11] Born, M., and Wolf, E., 1980, *Principles of Optics*, 6th edition (Oxford: Pergamon), Appendix V.

[12] Arnoldus, H. F., 2001, *J. opt. Soc. Am.* B, **18,** 547.

[13] Abramowitz, M., and Stegun, I. A., 1972, *Handbook of Mathematical Functions* (New York: Dover), p. 231.