Representation of the near-field, middle-field, and far-field electromagnetic Green's functions in reciprocal space

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Received July 10, 2000; revised manuscript received October 16, 2000

The electromagnetic field, generated by a source, has four typical components: the far field, the middle field, the near field, and the self-field. This decomposition is studied with the help of the dyadic Green's function for the electric field and its representation in reciprocal (**k**) space. The representations in **k** space involve three universal functions, which we call the $T(q)$ functions. Various representations of these functions are presented, and an interesting sum rule is derived. It is shown that the magnetic field can be split in a similar way, leading to a middle field and a far field only.

OCIS codes: 000.3860, 000.6800, 260.2110.

1. INTRODUCTION

Maxwell's equations govern the generation of electromagnetic radiation by charges and currents. In conventional approaches one derives a formal solution, such as a multipole expansion or an integral representation, and then develops an asymptotic expansion for the field far away from the source. For microscopic sources this far field used to be the only part of the radiation amenable to experimental observation. With recent progress in nanoscale technology and light detection with microscopic optical fiber tips, also the field close to the source can be measured in detail.[1–3] Electric fields can be observed up to within approximately a wavelength of resolution in the vicinity of a source. This so-called near-field optics has attracted great experimental attention, but it appears that little progress has been made in the theoretical investigations of these near fields. This seems partially because the response of the fiber tip, e.g., the detector, has to be taken into account as well. This leads to complicated geometries, and such configurations have been studied mainly numerically. Here the first step of a different approach to the evaluation of optical near fields is presented, based on a representation of these fields in reciprocal space.

2. INTEGRAL SOLUTION OF MAXWELL'S EQUATIONS

The time dependence of the electric field $\mathbf{E}(\mathbf{r}, t)$ can be Fourier transformed according to

$$\hat{\mathbf{E}}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} \mathrm{d}t \mathbf{E}(\mathbf{r}, t) \exp(i\omega t). \tag{2.1}$$

Since $\mathbf{E}(\mathbf{r}, t)$ is real we have $\hat{\mathbf{E}}(\mathbf{r}, -\omega) = \hat{\mathbf{E}}(\mathbf{r}, \omega)^*$, and therefore we need to consider only $\omega>0$. The inverse of Eq. (2.1) then becomes

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{\pi} \mathrm{Re} \int_0^{\infty} \mathrm{d}\omega \hat{\mathbf{E}}(\mathbf{r}, \omega) \exp(-i\omega t). \tag{2.2}$$

The magnetic field $\mathbf{B}(\mathbf{r}, t)$, the charge density $\rho(\mathbf{r}, t)$, and the current density $\mathbf{j}(\mathbf{r}, t)$ transform similarly. From here on we shall suppress the ω dependence and simply write $\hat{\mathbf{E}}(\mathbf{r})$, etc.

In the frequency domain, Maxwell's equations are

$$\nabla \cdot \hat{\mathbf{E}} = \hat{\rho}/\epsilon_0, \tag{2.3}$$

$$\nabla \times \hat{\mathbf{E}} = i\omega \hat{\mathbf{B}}, \tag{2.4}$$

$$\nabla \cdot \hat{\mathbf{B}} = 0, \tag{2.5}$$

$$\nabla \times \hat{\mathbf{B}} = -\frac{i\omega}{c^2} \hat{\mathbf{E}} + \mu_0 \hat{\mathbf{j}}. \tag{2.6}$$

The charge and current densities are related by the continuity equation $\nabla \cdot \hat{\mathbf{j}} = i\omega\hat{\rho}$. It can then be verified by substitution that a solution is given by

$$\hat{\mathbf{E}}(\mathbf{r}) = \frac{i\omega\mu_0}{4\pi} \int \mathrm{d}^3\mathbf{r}' g(\mathbf{r} - \mathbf{r}') \hat{\mathbf{j}}(\mathbf{r}') + \frac{i\omega\mu_0}{4\pi k_0^2} \nabla \left[\nabla \cdot \int \mathrm{d}^3\mathbf{r}' g(\mathbf{r} - \mathbf{r}') \hat{\mathbf{j}}(\mathbf{r}') \right], \tag{2.7}$$

$$\hat{\mathbf{B}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \nabla \times \int \mathrm{d}^3\mathbf{r}' g(\mathbf{r} - \mathbf{r}') \hat{\mathbf{j}}(\mathbf{r}'), \tag{2.8}$$

where $g(\mathbf{r})$ is the scalar Green's function, defined by

$$g(\mathbf{r}) = \exp(ik_0 r)/r, \tag{2.9}$$

and $k_0 = \omega/c$. We shall also suppress the ω dependence of Green's functions in the notation. In order to verify this solution, we need only the identity

$$(\nabla^2 + k_0^2) \int d^3\mathbf{r}' g(\mathbf{r} - \mathbf{r}')\hat{\mathbf{j}}(\mathbf{r}') = -4\pi\hat{\mathbf{j}}(\mathbf{r}). \quad (2.10)$$

The Green's function $g(\mathbf{r})$ is not defined for $r = 0$, and in the integrals above, it is understood that a small sphere with radius δ around the point $\mathbf{r}$ is excluded from the integration range. In the end we then take the limit $\delta \to 0$. As a consequence, care should be exercised in differentiating such integrals, as in Eqs. (2.7), (2.8), and (2.10). Operation of ∇ on these integrals not only affects the $\mathbf{r}$ dependence of the Green's function in the integrand, it also moves the sphere. This might lead to an extra term when moving the differential operators under the integral sign.[4,5] It can be shown that for the integral in Eq. (2.7) we get

$$\nabla\left[\nabla \cdot \int d^3\mathbf{r}' g(\mathbf{r} - \mathbf{r}')\hat{\mathbf{j}}(\mathbf{r}')\right] = -\frac{4\pi}{3}\hat{\mathbf{j}}(\mathbf{r}) + \int d^3\mathbf{r}' \nabla\{\nabla \cdot [g(\mathbf{r} - \mathbf{r}')\hat{\mathbf{j}}(\mathbf{r}')]\}. \quad (2.11)$$

Here, the first term on the right-hand side is due to moving the sphere. In Eq. (2.8) the curl can be moved under the integral sign without any additional term appearing. In Eq. (2.10) the right-hand side of the equation is due to moving the sphere when ∇^2 acts on the integral, as can be seen from $(\nabla^2 + k_0^2)[g(\mathbf{r} - \mathbf{r}')\hat{\mathbf{j}}(\mathbf{r}')] = 0$ for $\mathbf{r} \neq \mathbf{r}'$, e.g., everywhere on the integration region.

3. DYADIC GREEN'S FUNCTION

It is advantageous to rewrite the solution from the previous section and bring it in dyadic form. To this end, we notice the identity

$$\nabla\{\nabla \cdot [g(\mathbf{r} - \mathbf{r}')\hat{\mathbf{j}}(\mathbf{r}')]\} = [\nabla\nabla g(\mathbf{r} - \mathbf{r}')] \cdot \hat{\mathbf{j}}(\mathbf{r}'), \quad (3.1)$$

where $\nabla\nabla g(\mathbf{r} - \mathbf{r}')$ is a dyadic operator. Both integrals in Eq. (2.7) can then be combined, and the solution takes the form

$$\hat{\mathbf{E}}(\mathbf{r}) = -\frac{i}{3\epsilon_0\omega}\hat{\mathbf{j}}(\mathbf{r}) + \frac{i\omega\mu_0}{4\pi}\int d^3\mathbf{r}' \overleftrightarrow{d}(\mathbf{r} - \mathbf{r}') \cdot \hat{\mathbf{j}}(\mathbf{r}'). \quad (3.2)$$

Here, $\overleftrightarrow{d}(\mathbf{r})$ is defined as

$$\overleftrightarrow{d}(\mathbf{r}) = \left(\overleftrightarrow{I} + \frac{1}{k_0^2}\nabla\nabla\right)g(\mathbf{r}), \quad (3.3)$$

and $\overleftrightarrow{I}$ stands for the unit dyad. For later reference, $\overleftrightarrow{d}(\mathbf{r})$ is given explicitly by

$$\overleftrightarrow{d}(\mathbf{r}) = \left(1 + \frac{i}{k_0 r} - \frac{1}{k_0^2 r^2}\right)\overleftrightarrow{I} g(\mathbf{r}) + \left(-1 - \frac{3i}{k_0 r} + \frac{3}{k_0^2 r^2}\right)\hat{\mathbf{r}}\hat{\mathbf{r}} g(\mathbf{r}), \quad (3.4)$$

with $\hat{\mathbf{r}}$ the unit vector into the $\mathbf{r}$ direction. Solution (3.2) is often used as the starting point in near-field or nanoscale calculations.[6,7] The first term on the right-hand side of Eq. (3.2), owing to moving the sphere, is called the self-field since it produces an electric field that is directly proportional to the current density at the same location. In classical texts, this term is often omitted[8] since it yields no contribution to the field outside the source region, and in particular it has no effect on the far field. From a mathematical point of view, without this self-field contribution, the $\hat{\mathbf{E}}$ field would not rigorously satisfy Maxwell's equations.[9] As we shall see below, this self-field contributes in an essential way to the representation of the Green's function in reciprocal space. Also, when the transverse and longitudinal components of the near field are considered separately, this self-field yields a nonvanishing contribution outside the source and consequently can not be neglected.[10]

Solution (3.2) can be written in a more compact way by introducing the following dyadic operator:

$$\overleftrightarrow{g}(\mathbf{r}) = -\frac{4\pi}{3k_0^2}\delta(\mathbf{r})\overleftrightarrow{I} + \overleftrightarrow{d}(\mathbf{r}). \quad (3.5)$$

Then Eq. (3.2) can be written as

$$\hat{\mathbf{E}}(\mathbf{r}) = \frac{i\omega\mu_0}{4\pi}\int d^3\mathbf{r}' \overleftrightarrow{g}(\mathbf{r} - \mathbf{r}') \cdot \hat{\mathbf{j}}(\mathbf{r}'). \quad (3.6)$$

Here it is understood that the integral over the delta function is performed in the usual way, whereas for the integration over the singularity of $\overleftrightarrow{d}(\mathbf{r} - \mathbf{r}')$ at $\mathbf{r}' = \mathbf{r}$ we leave out the small sphere. The dyadic operator $\overleftrightarrow{g}(\mathbf{r})$ is referred to as the dyadic Green's function.

We can eliminate the magnetic field from Maxwell's equations, which gives

$$k_0^2\hat{\mathbf{E}} - \nabla \times (\nabla \times \hat{\mathbf{E}}) = -i\omega\mu_0\hat{\mathbf{j}} \quad (3.7)$$

For the electric field. It follows by inspection, and with the help of Eq. (2.10), that expression (2.7) for $\hat{\mathbf{E}}(\mathbf{r})$ satisfies Eq. (3.7). If we then formally let $\overleftrightarrow{g}(\mathbf{r})$ be the solution of

$$k_0^2\overleftrightarrow{g}(\mathbf{r}) - \nabla \times [\nabla \times \overleftrightarrow{g}(\mathbf{r})] = -4\pi\delta(\mathbf{r})\overleftrightarrow{I}, \quad (3.8)$$

which can be checked with Eq. (3.3) for $\mathbf{r} \neq 0$, then solution (3.6) for $\hat{\mathbf{E}}(\mathbf{r})$ follows by superposition. This justifies Eq. (3.8) for the dyadic Green's function.

4. RECIPROCAL SPACE

An extremely useful representation of the fields and the Green's functions arises if one Fourier transforms with respect to the spatial dependence. This representation in reciprocal space, which we shall indicate as $\mathbf{k}$ space, has widespread applications. For instance, it leads to (although it is not identical to) the angular-spectrum representation of the radiation field,[11] which can be applied to obtain the asymptotic form of the field in the radiation zone (the far field).[12] Another example is the quantization of the electromagnetic field in Coulomb gauge, which proceeds through a transformation to reciprocal space.[13,14] The transform $F(\mathbf{k})$ of an arbitrary function $f(\mathbf{r})$ is defined as

$$F(\mathbf{k}) = \int d^3\mathbf{r} f(\mathbf{r})\exp(-i\mathbf{k} \cdot \mathbf{r}), \quad (4.1)$$

with inverse

$$f(\mathbf{r}) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathbf{k}\, F(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r}). \quad (4.2)$$

We indicate such a transform pair as $f(\mathbf{r}) \leftrightarrow F(\mathbf{k})$.

The transform of the scalar Green's function $g(\mathbf{r})$ is denoted by $G(\mathbf{k})$, and with Eq. (4.1) this is

$$G(\mathbf{k}) = \int \mathrm{d}^3\mathbf{r}\exp(-i\mathbf{k}\cdot\mathbf{r})\frac{\exp(ik_0 r)}{r}. \quad (4.3)$$

In order to evaluate this integral we use spherical coordinates and take the polar axis along vector $\mathbf{k}$. Integration over the angles θ and ϕ then yields

$$G(\mathbf{k}) = \frac{2\pi i}{k}\int_0^\infty \mathrm{d}r\{\exp[i(k_0 - k)r] - \exp[i(k_0 + k)r]\}, \quad (4.4)$$

and this integral does not exist in the upper limit. In order to remedy this problem, we add a small positive imaginary part to k_0, after which we obtain

$$G(\mathbf{k}) = \frac{4\pi}{k^2 - k_0^2 - i\epsilon}, \quad (4.5)$$

with $\epsilon\downarrow 0$. It should then be verified that this construction with ϵ indeed gives the correct Green's function, when transformed back with Eq. (4.2). To this end, we take spherical coordinates in $\mathbf{k}$ space, such that the polar axis is along vector $\mathbf{r}$. After integrating over the angles we find

$$g(\mathbf{r}) = -\frac{i}{\pi r}\int_0^\infty \mathrm{d}k\frac{k}{k^2 - k_0^2 - i\epsilon}[\exp(ikr) - \exp(-ikr)]. \quad (4.6)$$

This can be rewritten as

$$g(\mathbf{r}) = -\frac{i}{\pi r}\int_{-\infty}^\infty \mathrm{d}k\frac{k}{k^2 - k_0^2 - i\epsilon}\exp(ikr). \quad (4.7)$$

Because of the ϵ, the poles in the complex k plane at $k = k_0$ and $k = -k_0$ have just moved off the real axis. With contour integration and the residue theorem we then obtain $g(\mathbf{r}) = \exp(ik_0 r)/r$. This shows that the construction with ϵ indeed yields the correct retarded Green's function.

In order to find the $\mathbf{k}$ space representation of the dyadic Green's function, we start from Eq. (2.7). Let $\hat{\mathbf{E}}(\mathbf{r}) \leftrightarrow \hat{\boldsymbol{E}}(\mathbf{k})$ and $\hat{\mathbf{j}}(\mathbf{r}) \leftrightarrow \hat{\mathbf{J}}(\mathbf{k})$. With $\nabla \leftrightarrow i\mathbf{k}$ and the convolution theorem, Eq. (2.7) transforms into

$$\hat{\boldsymbol{E}}(\mathbf{k}) = \frac{i\omega\mu_0}{4\pi}\left\{\hat{\mathbf{J}}(\mathbf{k}) - \frac{1}{k_0^2}\mathbf{k}[\mathbf{k}\cdot\hat{\mathbf{J}}(\mathbf{k})]\right\}G(\mathbf{k}), \quad (4.8)$$

and this can be written as

$$\hat{\boldsymbol{E}}(\mathbf{k}) = \frac{i\omega\mu_0}{4\pi}\vec{G}(\mathbf{k})\cdot\hat{\mathbf{J}}(\mathbf{k}), \quad (4.9)$$

provided we set

$$\vec{G}(\mathbf{k}) = G(\mathbf{k})\left(\vec{I} - \frac{1}{k_0^2}\mathbf{k}\mathbf{k}\right). \quad (4.10)$$

This is the $\mathbf{k}$ space representation of the dyadic Green's function, since Eq. (4.9) is just the spatial transform of solution (3.6).

5. SPLITTING OF THE FIELD

The Green's function $\vec{g}(\mathbf{r})$ splits naturally into four distinctive parts, regarding the r dependence. The first term in Eq. (3.5) is a delta function, and we call this the self-field (SF) part of $\vec{g}(\mathbf{r})$. Then $\vec{d}(\mathbf{r})$ in Eq. (3.4) has a $1/r^3$ part, a $1/r^2$ part, and a $1/r$ part, and these terms are indicated as the near-field (NF), middle-field (MF) and far-field (FF) components of the Green's function. So we have

$$\vec{g}(\mathbf{r})_{\mathrm{SF}} = -\frac{4\pi}{3k_0^2}\delta(\mathbf{r})\vec{I}, \quad (5.1)$$

$$\vec{g}(\mathbf{r})_{\mathrm{NF}} = -\frac{1}{k_0^2 r^3}(\vec{I} - 3\hat{\mathbf{r}}\hat{\mathbf{r}})\exp(ik_0 r), \quad (5.2)$$

$$\vec{g}(\mathbf{r})_{\mathrm{MF}} = \frac{i}{k_0 r^2}(\vec{I} - 3\hat{\mathbf{r}}\hat{\mathbf{r}})\exp(ik_0 r), \quad (5.3)$$

$$\vec{g}(\mathbf{r})_{\mathrm{FF}} = \frac{1}{r}(\vec{I} - \hat{\mathbf{r}}\hat{\mathbf{r}})\exp(ik_0 r), \quad (5.4)$$

and $\vec{g}(\mathbf{r})$ is the sum of these four components. Then the field from Eq. (3.6) splits accordingly:

$$\hat{\mathbf{E}}(\mathbf{r}) = \sum_\alpha \hat{\mathbf{E}}(\mathbf{r})_{\alpha\mathrm{F}}, \quad (5.5)$$

with α = S, N, M, and F. Each field component is then given by

$$\hat{\mathbf{E}}(\mathbf{r})_{\alpha\mathrm{F}} = \frac{i\omega\mu_0}{4\pi}\int \mathrm{d}^3\mathbf{r}'\vec{g}(\mathbf{r}-\mathbf{r}')_{\alpha\mathrm{F}}\cdot\hat{\mathbf{j}}(\mathbf{r}'),$$
$$\alpha = \mathrm{S},\ \mathrm{N},\ \mathrm{M},\ \text{or}\ \mathrm{F}, \quad (5.6)$$

each of which has its typical r dependence. It should be noted that, for instance, the r dependence of the near field is not exactly $1/r^3$ because of the convolution with the source $\hat{\mathbf{j}}(\mathbf{r}')$. Only for a point source in $\mathbf{r} = 0$ will the r dependence be exactly $1/r^3$.

6. SPLITTING IN RECIPROCAL SPACE

We now seek the representations of the field components in $\mathbf{k}$ space. To this end we need to transform the Green's functions:

$$\vec{g}(\mathbf{r})_{\alpha\mathrm{F}} \leftrightarrow \vec{G}(\mathbf{k})_{\alpha\mathrm{F}},\quad \alpha = \mathrm{S},\ \mathrm{N},\ \mathrm{M},\ \text{or}\ \mathrm{F}, \quad (6.1)$$

after which the field components are

$$\hat{\boldsymbol{E}}(\mathbf{k})_{\alpha\mathrm{F}} = \frac{i\omega\mu_0}{4\pi}\vec{G}(\mathbf{k})_{\alpha\mathrm{F}}\cdot\hat{\mathbf{J}}(\mathbf{k}),\quad \alpha = \mathrm{S},\ \mathrm{N},\ \mathrm{M},\ \text{or}\ \mathrm{F}. \quad (6.2)$$

The simplest one to transform is the self-field since $\delta(\mathbf{r}) \leftrightarrow 1$. We immediately obtain from Eq. (5.1)

$$\vec{G}(\mathbf{k})_{\mathrm{SF}} = -\frac{4\pi}{3k_0^2}\vec{I}, \tag{6.3}$$

which is independent of $\mathbf{k}$.

For the near field we need to transform $\vec{g}(\mathbf{r})_{\mathrm{NF}}$ from Eq. (5.2). This requires the evaluation of the integral

$$\vec{G}(\mathbf{k})_{\mathrm{NF}} = -\frac{1}{k_0^2}\int \mathrm{d}^3\mathbf{r}\exp(-i\mathbf{k}\cdot\mathbf{r})\frac{\exp(ik_0r)}{r^3}(\vec{I} - 3\hat{\mathbf{r}}\hat{\mathbf{r}}). \tag{6.4}$$

We take again spherical coordinates in $\mathbf{r}$ space and integrate over the angles first. This gives

$$\vec{G}(\mathbf{k})_{\mathrm{NF}} = -\frac{4\pi}{k_0^2k}(\vec{I} - 3\hat{\mathbf{k}}\hat{\mathbf{k}})\int_0^\infty \mathrm{d}r\frac{\exp(ik_0r)}{r^3} \times\left[\frac{3}{k}\cos(kr) + \left(r - \frac{3}{k^2r}\right)\sin(kr)\right], \tag{6.5}$$

with $\hat{\mathbf{k}}$ the unit vector into the $\mathbf{k}$ direction. For $r \to 0$ we have $\sin(kr)/(kr) \to 1$, and it seems that the integrand diverges as $1/r^3$ for $r \to 0$. However, if we make a Taylor expansion of the integrand around $r = 0$, it appears that the integrand remains finite for $r \to 0$ because of a precise cancellation of diverging terms. Therefore the integral exists in the lower limit. After making a change of variables, the result can be written as

$$\vec{G}(\mathbf{k})_{\mathrm{NF}} = \frac{4\pi}{k_0^2}(\vec{I} - 3\hat{\mathbf{k}}\hat{\mathbf{k}})T(k_0/k)_{\mathrm{NF}}, \tag{6.6}$$

where we introduced the universal function

$$T(q)_{\mathrm{NF}} = -\int_0^\infty \frac{\mathrm{d}t}{t^3}\left[3\cos t + \left(t - \frac{3}{t}\right)\sin t\right]\exp(iqt), \tag{6.7}$$

needed for $q \geqslant 0$.

The Green's functions for the middle field and the far field can be obtained along similar lines. We find

$$\vec{G}(\mathbf{k})_{\mathrm{MF}} = \frac{4\pi}{k_0^2}(\vec{I} - 3\hat{\mathbf{k}}\hat{\mathbf{k}})T(k_0/k)_{\mathrm{MF}}, \tag{6.8}$$

$$\vec{G}(\mathbf{k})_{\mathrm{FF}} = (\vec{I} - \hat{\mathbf{k}}\hat{\mathbf{k}})G(\mathbf{k}) + \frac{4\pi}{k_0^2}(\vec{I} - 3\hat{\mathbf{k}}\hat{\mathbf{k}})T(k_0/k)_{\mathrm{FF}}, \tag{6.9}$$

with

$$T(q)_{\mathrm{MF}} = iq\int_0^\infty \frac{\mathrm{d}t}{t^2}\left[3\cos t + \left(t - \frac{3}{t}\right)\sin t\right]\exp(iqt), \tag{6.10}$$

$$T(q)_{\mathrm{FF}} = q^2\int_0^\infty \frac{\mathrm{d}t}{t}\left(\cos t - \frac{\sin t}{t}\right)\exp(iqt). \tag{6.11}$$

7. EVALUATION OF THE $T(q)$ INTEGRALS

The integrals representing the functions $T(q)_{\alpha\mathrm{F}}$ can be evaluated analytically. Let's start with $T(q)_{\mathrm{NF}}$, Eq. (6.7). The first thing to notice is that the integral cannot be split into three separate integrals, since these would each diverge in the lower limit. It is only the combination of the three that exists. The method to be followed here is to integrate by parts, and reduce the integral to a standard integral. In order to do so one has to keep the lower limit finite, say δ, and then in the end take the limit $\delta \to 0$. Integration by parts two times yields the intermediate result

$$T(q)_{\mathrm{NF}} = \frac{1}{3} - iq\int_0^\infty \frac{\mathrm{d}t}{t^3}(t\cos t - \sin t)\exp(iqt), \tag{7.1}$$

which we use in Appendix A, and two more times gives the representation

$$T(q)_{\mathrm{NF}} = \frac{1}{3} - \frac{1}{2}q^2 - \frac{iq}{2}(q^2 - 1)\int_0^\infty \frac{\mathrm{d}t}{t}\exp(iqt)\sin t. \tag{7.2}$$

This last integral is tabulated, and the final result for $T(q)_{\mathrm{NF}}$ becomes

$$T(q)_{\mathrm{NF}} = \frac{1}{3} - \frac{1}{2}q^2 + \frac{1}{4}q(q^2 - 1)\ln\left|\frac{1+q}{1-q}\right| - \frac{i\pi}{4}\begin{cases} q(q^2 - 1), & 0 \leqslant q < 1 \\ 0, & q > 1\end{cases}. \tag{7.3}$$

From the integral representations (6.7) and (6.10) we observe the relation

$$T(q)_{\mathrm{MF}} = -q\frac{\mathrm{d}}{\mathrm{d}q}T(q)_{\mathrm{NF}}, \tag{7.4}$$

which gives immediately with Eq. (7.3),

$$T(q)_{\mathrm{MF}} = \frac{3}{2}q^2 - \frac{1}{4}q(3q^2 - 1)\ln\left|\frac{1+q}{1-q}\right| + \frac{i\pi}{4}\begin{cases} q(3q^2 - 1), & 0 \leqslant q < 1 \\ 0, & q > 1\end{cases}. \tag{7.5}$$

Then we use representation (7.1) for $T(q)_{\mathrm{NF}}$ in Eq. (7.4), and this gives

$$T(q)_{\mathrm{MF}} = iq\int_0^\infty \frac{\mathrm{d}t}{t^3}(t\cos t - \sin t)\exp(iqt) - q^2\int_0^\infty \frac{\mathrm{d}t}{t^2}(t\cos t - \sin t)\exp(iqt). \tag{7.6}$$

These integrals are the same as the ones appearing in Eqs. (7.1) and (6.11). Therefore we find the simple relation

$$T(q)_{\mathrm{NF}} + T(q)_{\mathrm{MF}} + T(q)_{\mathrm{FF}} = \tfrac{1}{3}, \tag{7.7}$$

and from this

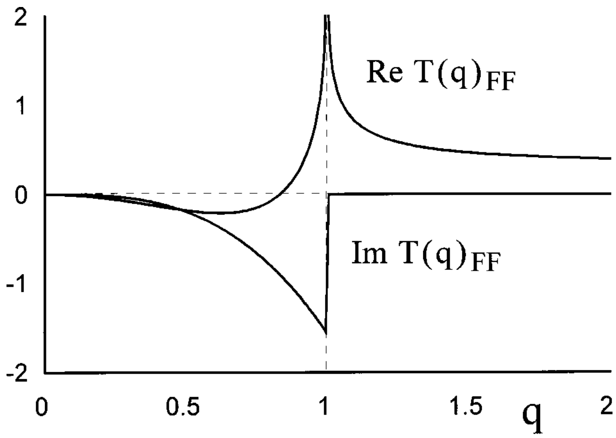

Fig. 1. Real and imaginary parts of the function $T(q)_{\mathrm{FF}}$.

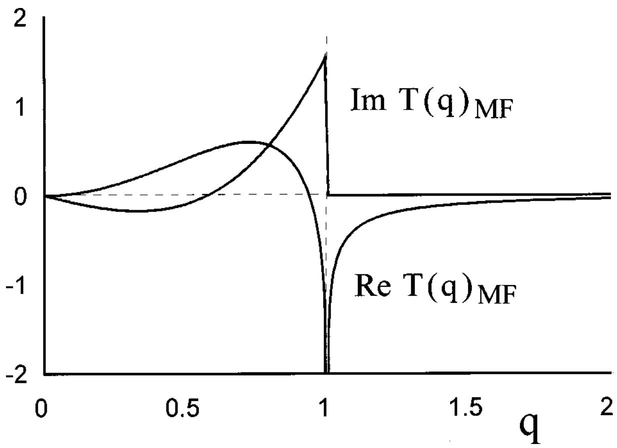

Fig. 2. Real and imaginary parts of the function $T(q)_{\mathrm{MF}}$.

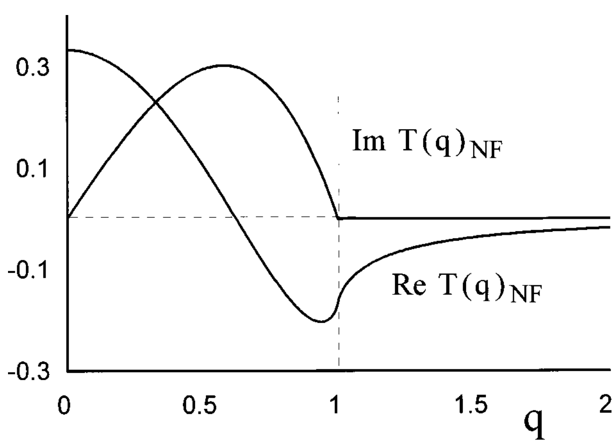

Fig. 3. Real and imaginary parts of the function $T(q)_{\mathrm{NF}}$.

$$T(q)_{\mathrm{FF}} = -q^2 + \frac{1}{2}q^3 \ln\left|\frac{1+q}{1-q}\right| - \frac{i\pi}{4}\begin{cases} 2q^3, & 0 \leqslant q < 1 \\ 0, & q > 1 \end{cases}. \tag{7.8}$$

The functions $T(q)_{\alpha\mathrm{F}}$ are shown in Figs. 1–3. For $q = 0$ we have $T(0)_{\mathrm{FF}} = T(0)_{\mathrm{MF}} = 0$, but $T(0)_{\mathrm{NF}} = 1/3$. For large q we have $T(\infty)_{\mathrm{MF}} = T(\infty)_{\mathrm{NF}} = 0$, and $T(\infty)_{\mathrm{FF}} = 1/3$, as can be found by Taylor expansion in $1/q$. For $q = 1$ the integrals have to be considered separately, since they can be discontinuous or diverging. We find $T(1)_{\mathrm{NF}} = -1/6$, $T(1)_{\mathrm{MF}} = -\infty + i\pi/4$, and $T(1)_{\mathrm{FF}} = \infty - i\pi/4$.

8. MORE ON THE $T(q)$ FUNCTIONS

From Eqs. (6.6), (6.8), and (6.9) and the sum rule (7.7) we find

$$\vec{G}(\mathbf{k})_{\mathrm{NF}} + \vec{G}(\mathbf{k})_{\mathrm{MF}} + \vec{G}(\mathbf{k})_{\mathrm{FF}} = (\vec{I} - \hat{\mathbf{k}}\hat{\mathbf{k}})G(\mathbf{k}) + \frac{4\pi}{3k_0^2}(\vec{I} - 3\hat{\mathbf{k}}\hat{\mathbf{k}}). \tag{8.1}$$

Adding the self-field from Eq. (6.3) gives

$$\sum_{\alpha} \vec{G}(\mathbf{k})_{\alpha\mathrm{F}} = (\vec{I} - \hat{\mathbf{k}}\hat{\mathbf{k}})G(\mathbf{k}) - \frac{4\pi}{k_0^2}\hat{\mathbf{k}}\hat{\mathbf{k}}, \tag{8.2}$$

and when we combine the terms, using Eq. (4.5), the right-hand side becomes $\vec{G}(\mathbf{k})$, as it should be. This clearly shows that the self-field cannot be omitted in a $\mathbf{k}$ space representation of the dyadic Green's function. Furthermore, for consistency we have to verify that when we transform the $\vec{G}(\mathbf{k})_{\alpha\mathrm{F}}$'s back to $\mathbf{r}$ space with the inverse transform (4.2), we indeed recover the functions $\vec{g}(\mathbf{r})_{\alpha\mathrm{F}}$. This is shown in Appendix A.

The $T(q)$ functions can be represented in a more compact form by considering q as a complex variable, although restricted to the range $0 \leqslant q < \infty$. They are

$$T(q)_{\mathrm{NF}} = \frac{1}{3} - \frac{1}{2}q^2 - \frac{1}{4}q(q^2 - 1)\ln\frac{q-1}{q+1}, \tag{8.3}$$

$$T(q)_{\mathrm{MF}} = \frac{3}{2}q^2 + \frac{1}{4}q(3q^2 - 1)\ln\frac{q-1}{q+1}, \tag{8.4}$$

$$T(q)_{\mathrm{FF}} = -q^2 - \frac{1}{2}q^3 \ln\frac{q-1}{q+1}. \tag{8.5}$$

For $\ln(z)$ with z complex we take the cut in the complex plane just below the negative real axis. For $\ln[(q-1)/(q+1)]$ this puts the cut in the q plane from $q = -1$ to $q = 1$, just below the real axis. For $0 \leqslant q < 1$ this gives the logarithm an imaginary part of $i\pi$.

Both the $\vec{G}(\mathbf{k})_{\mathrm{NF}}$ and the $\vec{G}(\mathbf{k})_{\mathrm{MF}}$ are proportional to the corresponding $T(q)$ functions, and both have the same dyadic form. The $\vec{G}(\mathbf{k})_{\mathrm{FF}}$, however, has an additional contribution. By inverse transform we find that the part proportional to the $T(q)_{\mathrm{FF}}$ transforms as (see Appendix A)

$$\frac{1}{k_0^2 r^3}(\vec{I} - 3\hat{\mathbf{r}}\hat{\mathbf{r}})[(1 - ik_0 r)\exp(ik_0 r) - 1] \leftrightarrow \frac{4\pi}{k_0^2}(\vec{I} - 3\hat{\mathbf{k}}\hat{\mathbf{k}})T(k_0/k)_{\mathrm{FF}}. \tag{8.6}$$

Apparently, this part only contains terms that go as r^{-3} and r^{-2}, whereas the total dyadic Green's function for the far field only has an r^{-1} term. Another peculiarity is the appearance of a nonretarded contribution (no $\exp(ik_0 r)$).

Evidently, the splitting of the right-hand side of Eq. (6.9) into two distinct terms has no physical significance in **r** space in the sense that it would split the far field into two terms with the typical r^{-1} behavior.

9. MAGNETIC FIELD

The solution for the magnetic field, Eq. (2.8), can be written as

$$\hat{\mathbf{B}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \mathrm{d}^3\mathbf{r}'\mathbf{c}(\mathbf{r}-\mathbf{r}')\times\hat{\mathbf{j}}(\mathbf{r}'), \tag{9.1}$$

with

$$\mathbf{c}(\mathbf{r}) = \nabla g(\mathbf{r}). \tag{9.2}$$

This vector field $\mathbf{c}(\mathbf{r})$ serves the same purpose for the magnetic field as the dyadic Green's function $\vec{g}(\mathbf{r})$ did for the electric field. Since

$$\mathbf{c}(\mathbf{r}) = \left(ik_0 - \frac{1}{r}\right)\hat{\mathbf{r}}g(\mathbf{r}), \tag{9.3}$$

we can immediately identify the field components, considering their r dependence. We see that the magnetic field does not have either a self-field or a near field. The middle and far fields are given by

$$\hat{\mathbf{B}}(\mathbf{r})_{\alpha\mathrm{F}} = \frac{\mu_0}{4\pi}\int \mathrm{d}^3\mathbf{r}'\mathbf{c}(\mathbf{r}-\mathbf{r}')_{\alpha\mathrm{F}}\times\hat{\mathbf{j}}(\mathbf{r}'), \quad \alpha = \mathrm{M},\ \mathrm{F}, \tag{9.4}$$

with

$$\mathbf{c}(\mathbf{r})_{\mathrm{MF}} = -\frac{1}{r}\hat{\mathbf{r}}g(\mathbf{r}), \tag{9.5}$$

$$\mathbf{c}(\mathbf{r})_{\mathrm{FF}} = ik_0\hat{\mathbf{r}}g(\mathbf{r}). \tag{9.6}$$

For the representation in reciprocal space, we let $\mathbf{c}(\mathbf{r}) \leftrightarrow \mathbf{C}(\mathbf{k})$. With Eq. (9.2) we then have

$$\mathbf{C}(\mathbf{k}) = i\mathbf{k}G(\mathbf{k}) \tag{9.7}$$

and $G(\mathbf{k})$ given by Eq. (4.5). With $\hat{\mathbf{B}}(\mathbf{r}) \leftrightarrow \hat{\boldsymbol{B}}(\mathbf{k})$, we have for the magnetic field in **k** space

$$\hat{\boldsymbol{B}}(\mathbf{k}) = \frac{\mu_0}{4\pi}\mathbf{C}(\mathbf{k})\times\hat{\mathbf{J}}(\mathbf{k}), \tag{9.8}$$

as follows from Eq. (9.1) and the convolution theorem.

Also in **k** space we can identify the middle- and far-field components of the magnetic field. Following the same procedure as in Section 6, we let

$$\mathbf{c}(\mathbf{r})_{\alpha\mathrm{F}} \leftrightarrow \mathbf{C}(\mathbf{k})_{\alpha\mathrm{F}}, \quad \alpha = \mathrm{M},\ \mathrm{F}, \tag{9.9}$$

so that the field components are

$$\hat{\boldsymbol{B}}(\mathbf{k})_{\alpha\mathrm{F}} = \frac{\mu_0}{4\pi}\mathbf{C}(\mathbf{k})_{\alpha\mathrm{F}}\times\hat{\mathbf{J}}(\mathbf{k}), \quad \alpha = \mathrm{M},\ \mathrm{F}. \tag{9.10}$$

It then remains to determine the functions $\mathbf{C}(\mathbf{k})_{\alpha\mathrm{F}}$, with

$$\mathbf{C}(\mathbf{k})_{\alpha\mathrm{F}} = \int \mathrm{d}^3\mathbf{r}\mathbf{c}(\mathbf{r})_{\alpha\mathrm{F}}\exp(-i\mathbf{k}\cdot\mathbf{r}). \tag{9.11}$$

Carrying out these integrations yields

$$\mathbf{C}(\mathbf{k})_{\mathrm{MF}} = -\frac{4\pi}{k_0^2}i\mathbf{k}T(k_0/k)_{\mathrm{FF}}, \tag{9.12}$$

$$\mathbf{C}(\mathbf{k})_{\mathrm{FF}} = i\mathbf{k}\left[G(\mathbf{k}) + \frac{4\pi}{k_0^2}T(k_0/k)_{\mathrm{FF}}\right]. \tag{9.13}$$

It is interesting to notice that for the splitting of the magnetic field we only need the $T(q)$ function for the electric far field. Also, it can be verified that the inverse transforms yield the correct results in **r** space.

10. RELATIONS FOR $\hat{E}_{\alpha\mathrm{F}}$ AND $\hat{B}_{\alpha\mathrm{F}}$

Maxwell's equations (2.3)–(2.6) hold for the total fields, but not for the components separately, in general. Nevertheless, there are some interesting relations for the various components. For instance, we see from Eqs. (9.10), (9.12), and (9.13) that $\hat{\boldsymbol{B}}(\mathbf{k})_{\alpha\mathrm{F}}$ is proportional to $\mathbf{k}\times\hat{\mathbf{J}}(\mathbf{k})$, and therefore we have $\mathbf{k}\cdot\hat{\boldsymbol{B}}(\mathbf{k})_{\alpha\mathrm{F}}$. In **r** space this is

$$\nabla\cdot\hat{\mathbf{B}}(\mathbf{r})_{\alpha\mathrm{F}} = 0, \tag{10.1}$$

e.g., the middle and far fields both satisfy Eq. (2.5) separately.

With Eqs. (6.2) and (6.9) we have

$$\hat{\boldsymbol{E}}(\mathbf{k})_{\mathrm{FF}} = \frac{i\omega\mu_0}{4\pi}G(\mathbf{k})\{\hat{\mathbf{J}}(\mathbf{k}) - \hat{\mathbf{k}}[\hat{\mathbf{k}}\cdot\hat{\mathbf{J}}(\mathbf{k})]\} + \frac{i\omega\mu_0}{k_0^2}T(k_0/k)_{\mathrm{FF}}\{\hat{\mathbf{J}}(\mathbf{k}) - 3\hat{\mathbf{k}}[\hat{\mathbf{k}}\cdot\hat{\mathbf{J}}(\mathbf{k})]\}, \tag{10.2}$$

and this gives

$$\frac{1}{\omega}\mathbf{k}\times\hat{\boldsymbol{E}}(\mathbf{k})_{\mathrm{FF}} = \frac{i\mu_0}{4\pi}\left[G(\mathbf{k}) + \frac{4\pi}{k_0^2}T(k_0/k)_{\mathrm{FF}}\right]\mathbf{k}\times\hat{\mathbf{J}}(\mathbf{k}). \tag{10.3}$$

With Eqs. (9.10) and (9.13) we see that this is exactly $\hat{\boldsymbol{B}}(\mathbf{k})_{\mathrm{FF}}$. In **r** space we therefore have

$$\nabla\times\hat{\mathbf{E}}(\mathbf{r})_{\mathrm{FF}} = i\omega\hat{\mathbf{B}}(\mathbf{r})_{\mathrm{FF}}, \tag{10.4}$$

showing that the far-field components satisfy Eq. (2.4). For the middle field we have

$$\frac{1}{\omega}\mathbf{k}\times\hat{\boldsymbol{E}}(\mathbf{k})_{\mathrm{MF}} = \frac{i\mu_0}{k_0^2}T(k_0/k)_{\mathrm{MF}}\mathbf{k}\times\hat{\mathbf{J}}(\mathbf{k}), \tag{10.5}$$

but $\hat{\boldsymbol{B}}(\mathbf{k})_{\mathrm{MF}}$ is with Eqs. (9.10) and (9.12),

$$\hat{\boldsymbol{B}}(\mathbf{k})_{\mathrm{MF}} = -\frac{i\mu_0}{k_0^2}T(k_0/k)_{\mathrm{FF}}\mathbf{k}\times\hat{\mathbf{J}}(\mathbf{k}), \tag{10.6}$$

and that is not the same as the right-hand side of (10.5). Therefore the middle field does not satisfy Eq. (2.4).

Another example is the divergence of the electric field. The representation in **k** space of $\nabla\cdot\hat{\mathbf{E}}(\mathbf{r})_{\mathrm{FF}}$ is with Eq. (10.2):

$$i\mathbf{k}\cdot\hat{\boldsymbol{E}}(\mathbf{k})_{\mathrm{FF}} = \frac{2}{\epsilon_0\omega}[\mathbf{k}\cdot\hat{\mathbf{J}}(\mathbf{k})]T(k_0/k)_{\mathrm{FF}}. \tag{10.7}$$

If we indicate by $\hat{R}(\mathbf{k})$ the transform of $\hat{\rho}(\mathbf{r})$ and use the continuity equation $\mathbf{k}\cdot\hat{\mathbf{J}} = \omega\hat{R}$, then Eq. (10.5) becomes

$$i\mathbf{k}\cdot\hat{\boldsymbol{E}}(\mathbf{k})_{\mathrm{FF}} = \frac{2}{\epsilon_0}\hat{R}(\mathbf{k})T(k_0/k)_{\mathrm{FF}}. \tag{10.8}$$

This same relation also holds for the middle field and the near field. For the self-field we obtain

$$i\mathbf{k}\cdot\hat{\boldsymbol{E}}(\mathbf{k})_{\mathrm{SF}} = \frac{1}{3\epsilon_0}\hat{R}(\mathbf{k}), \tag{10.9}$$

and with the sum rule (7.7) we then find for the divergence of the total field

$$\sum_{\alpha} i\mathbf{k}\cdot\hat{\boldsymbol{E}}(\mathbf{k})_{\alpha\mathrm{F}} = \frac{1}{\epsilon_0}\hat{R}(\mathbf{k}), \tag{10.10}$$

which is Eq. (2.3) in $\mathbf{k}$ space. This shows that each field component has a nonzero divergence, and their sum adds up to $\hat{\rho}/\epsilon_0$. Another interesting observation is the following. When $\hat{\rho}$ represents a localized source, then $\hat{\rho} = 0$ and $\nabla\cdot\hat{\mathbf{E}}(\mathbf{r}) = 0$ outside the source region. The inverse of Eq. (10.9) is

$$\nabla\cdot\hat{\mathbf{E}}(\mathbf{r})_{\mathrm{SF}} = \frac{1}{3\epsilon_0}\hat{\rho}(\mathbf{r}), \tag{10.11}$$

which also vanishes outside the source region. Therefore the sum of the near field, middle field, and far field also has a zero divergence outside the source. The three individual components, however, have a nonzero divergence outside the source.

11. CONCLUSIONS

We have studied the dyadic Green's function of electromagnetic theory from the point of view of its separation into near-field, middle-field, and far-field components. The representation of these various components in reciprocal space was obtained, and each component involves a universal $T(q)$ function. We have presented various integral representations of these functions, along with their explicit forms in Eqs. (8.3)–(8.5). It was shown that in $\mathbf{k}$ space it is essential to also account for the self-field Green's function; otherwise, the sum of the respective Green's functions would not combine into the total Green's function for the electric field. Then the magnetic field was split similarly, both in $\mathbf{r}$ space and $\mathbf{k}$ space. It turned out that for the magnetic field only the far-field $T(q)$ function enters the representation in reciprocal space.

APPENDIX A

In this appendix we show that when we apply the inverse transform (4.2) to the dyadic Green's functions $\vec{G}(\mathbf{k})_{\alpha\mathrm{F}}$, we indeed recover the functions $\vec{g}(\mathbf{r})_{\alpha\mathrm{F}}$. Starting with the far field, given by Eq. (6.9), we first consider the inverse of the term $(\vec{I} - \hat{\mathbf{k}}\hat{\mathbf{k}})G(\mathbf{k})$, with $G(\mathbf{k})$ given by Eq. (4.5). We use spherical coordinates in $\mathbf{k}$ space, with the polar axis along the fixed vector $\mathbf{r}$. When we integrate over the angles, we obtain for the inverse of $(\vec{I} - \hat{\mathbf{k}}\hat{\mathbf{k}})G(\mathbf{k})$ the representation

$$\begin{aligned}&\frac{1}{2\pi}\int_0^\infty \mathrm{d}k\,\frac{k^2}{k^2-k_0^2-i\epsilon}\\&\quad\times\Bigg\langle(\vec{I}+\hat{\mathbf{r}}\hat{\mathbf{r}})\frac{1}{ikr}[\exp(ikr)-\exp(-ikr)]\\&\quad+(\vec{I}-3\hat{\mathbf{r}}\hat{\mathbf{r}})\Bigg\{\left[\frac{1}{ikr}+\frac{2}{(ikr)^3}\right][\exp(ikr)\\&\quad-\exp(-ikr)]-\frac{2}{(ikr)^2}[\exp(ikr)+\exp(-ikr)]\Bigg\}\Bigg\rangle.\end{aligned} \tag{A1}$$

Here the terms with $\exp(-ikr)$ can be taken into account by extending the integration range to $-\infty$. We then obtain the representation

$$\begin{aligned}&\frac{1}{2\pi}\int_0^\infty \mathrm{d}k\,\frac{k^2\exp(ikr)}{k^2-k_0^2-i\epsilon}\Bigg\langle(\vec{I}+\hat{\mathbf{r}}\hat{\mathbf{r}})\frac{1}{ikr}\\&\quad+(\vec{I}-3\hat{\mathbf{r}}\hat{\mathbf{r}})\left[\frac{1}{ikr}-\frac{2}{(ikr)^2}+\frac{2}{(ikr)^3}\right]\Bigg\rangle.\end{aligned} \tag{A2}$$

The integrand has three first-order poles, located at $k = 0$ and $k = \pm(k_0^2 + i\epsilon)^{1/2}$. In order to evaluate this integral over k, we close the contour with a semicircle in the upper half of the complex k plane. Then the pole at $k = (k_0^2 + i\epsilon)^{1/2}$ is within the contour, the pole at $k = -(k_0^2 + i\epsilon)^{1/2}$ is outside the contour, and the pole at $k = 0$ is exactly on the contour. This last one should be taken as a principal-value integral, so we first go around this pole with a small semicircle in the upper half of the k plane and evaluate the contour integral with the residue theorem. From the result we then subtract the integral over the small semicircle. The final result is

$$\begin{aligned}&(\vec{I}-\hat{\mathbf{r}}\hat{\mathbf{r}})\frac{\exp(ik_0r)}{r}-\frac{1}{k_0^2r^3}(\vec{I}-3\hat{\mathbf{r}}\hat{\mathbf{r}})\\&\quad\times[(1-ik_0r)\exp(ik_0r)-1]\leftrightarrow(\vec{I}-\hat{\mathbf{k}}\hat{\mathbf{k}})G(\mathbf{k}).\end{aligned} \tag{A3}$$

Next we consider the inverse of $4\pi k_0^{-2}(\vec{I} - 3\hat{\mathbf{k}}\hat{\mathbf{k}})T(k_0/k)_{\mathrm{FF}}$. When we integrate first over the angles in $\mathbf{k}$ space, we obtain a result similar in structure as Eq. (A1). The subsequent integration over k, however, follows a different route. We set $u = kr$, which gives the transform pair

$$\frac{2}{\pi k_0^2r^3}(\vec{I}-3\hat{\mathbf{r}}\hat{\mathbf{r}})I(k_0r)_{\mathrm{FF}}\leftrightarrow\frac{4\pi}{k_0^2}(\vec{I}-3\hat{\mathbf{k}}\hat{\mathbf{k}})T(k_0/k)_{\mathrm{FF}}, \tag{A4}$$

where we introduced the auxiliary function

$$I(b)_{\mathrm{FF}} = \int_0^\infty \mathrm{d}u\,T(b/u)_{\mathrm{FF}}\left[\left(u-\frac{3}{u}\right)\sin u + 3\cos u\right]. \tag{A5}$$

In order to evaluate this integral, we use the integral representation (6.11) for the function $T(q)_{\mathrm{FF}}$. We make a change of integration variables from t to v according to

$v = bt/u$ and then insert this into Eq. (A5). Then we change the order of integration, and in the integral over u we set $p = v/b$. This leads to the following representation for $I(b)_{\mathrm{FF}}$:

$$I(b)_{\mathrm{FF}} = b^2 \int_0^\infty \frac{\mathrm{d}v}{v} \exp(iv) \int_0^\infty \frac{\mathrm{d}u}{u^2} \left[\left(u - \frac{3}{u} \right) \sin u + 3 \cos u \right] \left[\cos(pu) - \frac{1}{pu} \sin(pu) \right]. \quad \text{(A6)}$$

First we perform the integration over u, integrating by parts several times and keeping the lower limit finite. In the end we then let the lower limit approach zero. The result is

$$\int_0^\infty \frac{\mathrm{d}u}{u^2} \left[\left(u - \frac{3}{u} \right) \sin u + 3 \cos u \right] \times \left[\cos(pu) - \frac{1}{pu} \sin(pu) \right] = \begin{cases} \frac{1}{2} \pi p^2, & 0 \leq p < 1 \\ 0, & p > 1 \end{cases}. \quad \text{(A7)}$$

Then we set again $p = v/b$ and substitute this into Eq. (A6). Carrying out the v integration then yields

$$I(b)_{\mathrm{FF}} = \frac{\pi}{2} [(1 - ib)\exp(ib) - 1], \quad \text{(A8)}$$

and with relation (A4) this then gives the relation shown in Eq. (8.6). Finally, when we add Eq. (8.6) to relation (A3), the left-hand side is the far-field Green's function from Eq. (5.4).

For the middle field we need to invert $4\pi k_0^{-2}(\overleftrightarrow{I} - 3\hat{\mathbf{k}}\hat{\mathbf{k}}) T(k_0/k)_{\mathrm{MF}}$. This proceeds along the same lines as in the previous paragraph, leading to relations (A4) and (A5) with FF→MF. For $T(q)_{\mathrm{MF}}$ we use the representation (6.10), and instead of Eq. (A6) we are now left with

$$I(b)_{\mathrm{MF}} = ib^2 \int_0^\infty \frac{\mathrm{d}v}{v^2} \exp(iv) \int_0^\infty \frac{\mathrm{d}u}{u^2} \left[\left(u - \frac{3}{u} \right) \sin u + 3 \cos u \right] \times \left[\left(pu - \frac{3}{pu} \right) \sin(pu) + 3 \cos(pu) \right]. \quad \text{(A9)}$$

The integration over u here leads to a complication. When we split the integral in several terms, keeping the lower limit finite at first, then one of the integrals has as integrand $\sin(u)\sin(pu)$, and this integral does not exist in the upper limit. All others do, and there is no cancellation of terms. In fact, the combination of all others turns out to be identically zero. For this problematic term we keep the upper limit finite for the time being, and this gives

$$\int_0^\infty \frac{\mathrm{d}u}{u^2} \left[\left(u - \frac{3}{u} \right) \sin u + 3 \cos u \right] \left[\left(pu - \frac{3}{pu} \right) \sin(pu) + 3 \cos(pu) \right] = p \int_0^{u_{\max}} \mathrm{d}u \sin u \sin(pu). \quad \text{(A10)}$$

The integration variable u came from the substitution $u = kr$, so the upper limit is $u_{\max} = k_{\max} r$, with $k_{\max}$ the radius of a sphere in $\mathbf{k}$ space. In Eq. (A10) it is understood that eventually we take the upper limit to infinity. Rather than evaluating this integral, we keep this representation and substitute it into Eq. (A9). Then we integrate over v first, yielding

$$I(b)_{\mathrm{MF}} = -\frac{b}{4} \int_0^{u_{\max}} \mathrm{d}u \sin u \ln\left(\frac{u + b}{u - b} \right)^2 + \frac{ib\pi}{2} \int_0^{u_{\max}} \mathrm{d}u \sin u. \quad \text{(A11)}$$

The first integral exists for $u_{\max} \to \infty$, but a new problem arises. The integrand has a singularity at $u = b$, and the integral should be understood as a principal-value integral. We leave out the interval $b - \epsilon < u < b + \epsilon$, and in the end we take the limit $\epsilon \to 0$. Integration by parts gives

$$\int_0^\infty \mathrm{d}u \sin u \ln\left(\frac{u + b}{u - b} \right)^2 = 4 \int_0^\infty \mathrm{d}u \cos u \frac{1}{b^2 - u^2}, \quad \text{(A12)}$$

where we have taken the limit $\epsilon \to 0$ in the integrated part. The right-hand side is still a principal-value integral, and with contour integration we obtain

$$\int_0^\infty \mathrm{d}u \sin u \ln\left(\frac{u + b}{u - b} \right)^2 = 2\pi \sin b. \quad \text{(A13)}$$

With Eq. (A11) we then find

$$I(b)_{\mathrm{MF}} = \tfrac{1}{2} ib\pi [\exp(ib) - \cos(k_{\max} r)]. \quad \text{(A14)}$$

At this point we consider the limit $k_{\max} \to \infty$. Obviously, this limit does not exist in the strictest sense. However, every Green's function is eventually multiplied by some function of $\mathbf{r}$, representing the source of the field, followed by an integration over $\mathbf{r}$. For $k_{\max}$ large, the term $\cos(k_{\max} r)$ is very rapidly oscillating as a function of r and will integrate to zero. Therefore at this stage we simply leave this term out. It should be noted that such terms occur frequently in inverse transforms that do not exist in the strictest sense. The most notable example is the integral representation of the delta function. Then finally with relation (A4) we recover the Green's function for the middle field.

For the near field we need to consider $4\pi k_0^{-2}(\overleftrightarrow{I} - 3\hat{\mathbf{k}}\hat{\mathbf{k}}) T(k_0/k)_{\mathrm{NF}}$. Here we encounter a different type of problem. As found in Section 7, we have $T(0)_{\mathrm{NF}} = 1/3$, so that $T(k_0/k)_{\mathrm{NF}} \to 1/3$ for $k \to \infty$. For the inverse integral (4.2) to exist, it seems necessary that the function to be inverted goes to zero sufficiently fast for $k \to \infty$, and that is apparently not the case here. In particular, the inverse of a constant function is a delta function: $\delta(\mathbf{r}) \leftrightarrow 1$. As a solution, we first subtract the constant of 1/3 in $T(k_0/k)_{\mathrm{NF}}$, according to

$$\overleftrightarrow{g}(\mathbf{r})_{\mathrm{NF}} \leftrightarrow \frac{4\pi}{3k_0^2}(\overleftrightarrow{I} - 3\hat{\mathbf{k}}\hat{\mathbf{k}}) + \frac{4\pi}{k_0^2}(\overleftrightarrow{I} - 3\hat{\mathbf{k}}\hat{\mathbf{k}})[T(k_0/k)_{\mathrm{NF}} - 1/3]. \quad \text{(A15)}$$

Then we notice the transform pairs $\overleftrightarrow{\delta}(\mathbf{r}) = \delta(\mathbf{r})\overleftrightarrow{I} \leftrightarrow \overleftrightarrow{I}$ and $\overleftrightarrow{\delta}(\mathbf{r})^{\parallel} \leftrightarrow \hat{\mathbf{k}}\hat{\mathbf{k}}$, with $\overleftrightarrow{\delta}(\mathbf{r})$ and $\overleftrightarrow{\delta}(\mathbf{r})^{\parallel}$ the dyadic delta function and its longitudinal part, respectively. Since we have

$$\overleftrightarrow{\delta}(\mathbf{r})^{\parallel} = \frac{1}{3}\overleftrightarrow{\delta}(\mathbf{r}) + \frac{1}{4\pi r^3}(\overleftrightarrow{I} - 3\hat{\mathbf{r}}\hat{\mathbf{r}}), \quad \text{(A16)}$$

the first term on the right-hand side of relation (A15) transforms as

$$-\frac{1}{k_0^2 r^3}(\overleftrightarrow{I} - 3\hat{\mathbf{r}}\hat{\mathbf{r}}) \leftrightarrow \frac{4\pi}{3k_0^2}(\overleftrightarrow{I} - 3\hat{\mathbf{k}}\hat{\mathbf{k}}), \quad \text{(A17)}$$

which contains no delta function after all. The evaluation of the inverse of the second term goes just as for the far field and the middle field, except that we replace $T(k_0/k)_{\alpha\mathrm{F}}$ by $T(k_0/k)_{\mathrm{NF}} - 1/3$. Here we use representation (7.1), which already has the 1/3 split off. Without any further complications we then obtain

$$\frac{1}{k_0^2 r^3}(\overleftrightarrow{I} - 3\hat{\mathbf{r}}\hat{\mathbf{r}})[1 - \exp(ik_0 r)] \leftrightarrow \frac{4\pi}{k_0^2}(\overleftrightarrow{I} - 3\hat{\mathbf{k}}\hat{\mathbf{k}})[T(k_0/k)_{\mathrm{NF}} - 1/3], \quad \text{(A18)}$$

and the sum of relations (A17) and (A18) then gives the Green's function for the near field, Eq. (5.2).

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