OVERVIEW OF THE COMBINATORICS FUNCTION TECHNIQUE

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Abstract

This paper summarizes the combinatorics function technique leading to the solution of a multidimensional linear recurrence relation subject to a set of initial conditions.

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1. Multidimensional linear recurrence relations

A multidimensional linear recurrence relation (or partial difference equation) may be written in the form [1]:

$$B(\vec{M}) = \sum_{k=1}^{N} f_{A_k}(\vec{M}) B(\vec{M} - \vec{A}_k) \quad ; \quad \vec{M} \in R \quad . \tag{1}$$

The quantity $B(\vec{M})$ depends on the set of n variables $\{x_1, x_2, \ldots, x_n\}$ with which one associates a point, M, in an n-dimensional Euclidean space having for coordinates the same set of numbers. The position vector of that point is denoted $\vec{M}$, and the recurrence relation is valid for all points belonging to region R. One also refers to $B(\vec{M})$ as the quantity B evaluated at point M. Equation (1) is an (N + 1)-term linear recurrence relation, expressing that $B(\vec{M})$ is linearly related to N other values of B evaluated at points with position vectors $\{\vec{M} - \vec{A}_k ; k = 1, \ldots, N\}$. Coefficients $f_{A_k}(\vec{M})$ are labeled by the corresponding shifts $\vec{A}_k$ and may also depend on the evaluation point M. When these coefficients are independent of the evaluation point, the linear recurrence relation has constant coefficients. While Eq. (1) states that the recurrence is valid for points M in region R, in general the values of B in that region are not uniquely determined unless its values at some other points are specified. Assume that the values of B are known at "boundary" points, J_ℓ, corresponding to position vectors $\vec{J}_\ell$, according to:

$$B(\vec{J}_\ell) = \lambda_\ell \quad ; \quad \ell = 1, 2, \ldots \quad . \tag{2}$$

The boundary points form a *boundary* region $R_b = \{\vec{J}_\ell ;\ \ell = 1, 2, \ldots\}$ and the λ_ℓ's are called *initial values*. Consider the set of points R_J which can be reached following any number of displacements made of the elements of set A = $\{\vec{A}_k ; k = 1, \ldots, N\}$, and leaving any given boundary point, without encountering any of the other boundary points. A solution to Eq. (1) satisfying the boundary conditions (2) exists and is unique, if and only if $R \subset R_J$. If there is a boundary point which cannot be linked to any of the points in R without encountering another boundary point, then the solution does not depend on the initial value associated with this boundary point. If region R_b does not contain such boundary points, it is then called a *minimal* boundary. For all practical purposes, we will assume that $R = R_J$ and that R_b is a minimal boundary. Thus the solution of Eq. (1) satisfying the initial value conditions (2) is a linear combination of all the initial values listed in Eq. (2). We shall write this as [1]:

$$B(\vec{M}) = \sum_{\ell} \lambda_{\ell}\, C(\vec{J}_{\ell}, \vec{M}) \quad . \tag{3}$$

The quantity $C(\vec{J}_{\ell}, \vec{M})$ is called the *combinatorics function*. Its construction is based on all possible paths leaving the boundary point J_{ℓ} and reaching the evaluation point M by successive displacements belonging to set A, while avoiding all other boundary points. The construction of a combinatorics function is presented in the following paragraph.

A given path leaving J_{ℓ} and reaching M is identified by two labels, ω and q. The former label is the number of displacements in this path, and the latter is used to distinguish the various discrete paths having the same number of displacements ω. Consider a given path (ωq) with displacements $\vec{\delta}_1^q$, $\vec{\delta}_2^q$, ..., $\vec{\delta}_i^q$, ..., $\vec{\delta}_\omega^q$, made in this order, leaving J_{ℓ} and reaching M. Let S_i be the point on this path reached after the i^{th} displacement, $\vec{\delta}_i^q$. The position vector of that point is:

$$\vec{S}_i = \vec{J}_{\ell} + \sum_{j=0}^{i} \vec{\delta}_j^q \quad ; \quad i = 0,1,...,\omega \quad ; \quad \vec{S}_o = \vec{J}_{\ell} \quad ; \quad \vec{S}_{\omega} = \vec{M} \quad . \tag{4a}$$

In this equation we have introduced for convenience the nil displacement,

$$\vec{\delta}_0^q = \vec{0} \quad ; \quad \forall\ q \quad . \tag{4b}$$

On the (ωq)-path, point S_i is reached following displacements $\vec{\delta}_1^q$, $\vec{\delta}_2^q$, ..., $\vec{\delta}_i^q$. With this point one associates the quantity $f_{\delta_i^q}(\vec{S}_i)$, and with the (ωq)-path one associates the product

$$F_{\omega}^q(\vec{J}_{\ell}, \vec{M}) = \prod_{i=0}^{\omega} f_{\delta_i^q}(\vec{S}_i) \quad , \tag{5a}$$

where the coefficient associated with the nil displacement is unity:

$$f_{\delta_0^q}(\vec{S}) = f_0(\vec{S}) = 1 \quad , \quad \forall \vec{S} \quad . \tag{5b}$$

The combinatorics function is then given by:

$$C(\vec{J}_\ell, \vec{M}) = \sum_{\omega,q} F_\omega^q(\vec{J}_\ell, \vec{M}) = \sum_{\omega,q} \left\{ \prod_{i=0}^{\omega} f_{\delta_0^q}(\vec{S}) \right\} \quad . \tag{6}$$

By adding to the right-hand side of Eq. (1) a term, $I(\vec{M})$, which may depend on the evaluation point, the recurrence relation becomes inhomogeneous,

$$B(\vec{M}) = I(\vec{M}) + \sum_{k=1}^{N} f_{A_k}(\vec{M})\, B(\vec{M} - \vec{A}_k) \quad ; \quad \vec{M} \in R \tag{7}$$

The solution of Eq. (7) satisfying the boundary conditions of Eq. (2) is given by [2]:

$$B(\vec{M}) = \sum_{\ell} \lambda_\ell\, C(\vec{J}_\ell, \vec{M}) + \sum_{\vec{L} \in R} I(\vec{L})\, C(\vec{L}, \vec{M}) \quad . \tag{8}$$

The combinatorics function $C(\vec{L}, \vec{M})$ is constructed based on all possible paths (made of displacements belonging to set A) connecting point L and the evaluation point M and avoiding the boundary points. Equivalently, one may also view the solution of the inhomogeneous equation as the sum of two terms: the solution of the homogeneous equation and the particular solution of the inhomogeneous equation corresponding to all the initial values $\lambda_i = 0$.

2. One-dimensional two-term recurrence relations

A. General method

With appropriate scaling, a two-term recurrence on a function u(x) subject to the initial value condition,

$$u(x_o) = \lambda \quad , \tag{9a}$$

may be written as:

$$u(x) = p(x)\, u(x-1) + q(x) \quad , \tag{9b}$$

where p(x) and q(x) are known functions. The solution of Eq. (9) is well known [3] and may be obtained by the method of “variation of parameters,” which relies on the knowledge of the solution $u_1(x)$ of the homogeneous equation, and then searching for v(x) such that $u(x) = v(x)\, u_1(x)$. A special solution may also be obtained as an ascending continued fraction for $x > x_o$ [3]. The combinatorics function technique may also be used to recover the same result [4]. For definiteness, let us assume that $x > x_o$. Set A contains only one element which is the displacement by one unit in the direction of the positive x-axis. The function associated with this displacement is $f_1(x) = p(x)$, and the inhomogeneous term is $I(x) = q(x)$. Region R for which the solution is obtained uniquely in terms of λ is the one containing the points with coordinate x such that $x - x_o$ is a positive integer, say n. There exists only one path connecting any two points of coordinates y and x for which $x - y = m$ is a positive integer. Thus the one-dimensional combinatorics function associated with this path is

$$C(y,x) = \prod_{j=1}^{m} p(y+j) \quad , \quad C(y,y) = 1 \quad , \tag{10}$$

and the solution of Eq. (9) follows:

$$u(x) = \lambda C(x_o, x) + \sum_{j=1}^{m} q(x_o + j)\, C(x_o + j, x) \quad . \tag{11}$$

The ascending continued fraction solution is also obtained as a special case of the combinatorics function technique, when replacing Eq. (9b) by:

$$u(x) = \frac{u(x+1)}{p(x+1)} + \frac{q(x+1)}{p(x+1)} \quad ; \quad x > x_o \quad . \tag{12}$$

Set A has only one element corresponding to the unit displacement in the direction of the negative x-axis, and the function associated with this displacement is $f_{-1}(x) = 1/p(x+1)$. The inhomogeneous term is $I(x) = q(x+1)/p(x+1)$. A minimal boundary region associated with region R must contain one point whose coordinate y satisfying the inequality $y > x > x_o$ with the constraint that $x - y = n$ is a positive integer. The combinatorics function is denoted in this case as $C^*(y, x)$ to distinguish it from the one above, and it is given by:

$$C^*(y,x) = \prod_{j=1}^{m} \frac{1}{p(x+j)} \quad ; \quad C^*(y,y) = 1 \quad . \tag{13}$$

The general solution of Eq. (12) with the boundary point being at an arbitrary location ($y > x > x_o$) then follows:

$$u(x) = u(y)C^*(y,x) + \sum_{k=0}^{n} \frac{q(x+k+1)}{p(x+k+1)} C^*(x+k,x) \quad . \tag{14}$$

The ascending continued fraction solution given by Milne-Thomson [3,5] is a special case of Eq. (14) where the boundary point is at infinity and $u(\infty) = 0$. Then Eq. (14) is equivalent to:

$$u(x) = \cfrac{q(x+1) + \cfrac{q(x+2) + \cfrac{q(x+3) + \cfrac{q(x+4) + \cdot^{\cdot^{\cdot}}}{p(x+4)}}{p(x+3)}}{p(x+2)}}{p(x+1)} \quad . \tag{15}$$

B. Bernouilli and Euler polynomials

Let $B_n(x)$ and $E_n(x)$ be the Bernouilli and Euler polynomials, respectively. They are defined through the following generating functions [6]:

$$\frac{t\, e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad , \quad |t| < 2\pi \quad ; \tag{16}$$

$$\frac{2\, e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad , \quad |t| < \pi \quad . \tag{17}$$

They also satisfy the two-term recurrence relations:

$$B_{n+1}(x+1) = B_{n+1}(x) + (n+1)x^n \quad ; \tag{18}$$

$$E_n(x) = -E_n(x) + 2x^n \quad . \tag{19}$$

We choose the boundary point arbitrarily at x_o such that

$$x_o = x - [x] \quad , \tag{20}$$

where [x] indicates the integer part of x, which in turn shows that x_o is in the range $0 \le x_o < 1$. The initial values for recurrence relations (18) and (19) are $B_{n+1}(x_o)$ and $E_n(x_o)$, respectively. In the Bernouilli case, we have p(x + 1) = 1 and $q(x + 1) = (n + 1)\, x^n$, and Eq. (11) implies [4]:

$$\sum_{k=1}^{[x]} (k + x_o)^n = \frac{B_{n+1}(x) - B_{n+1}(x_o)}{n+1} \quad . \tag{21}$$

In the Euler case, $p(x + 1) = -1$, $q(x + 1) = 2\, x^n$ and Eq. (11) implies [4]:

$$\sum_{k=1}^{[x]} (-1)^{[x]-k} (k + x_o)^n = \frac{E_n(x) + (-1)^{[x]} E_n(x_o)}{2} \quad . \tag{22}$$

These results hold for $[x] \ge 1$ with n a positive integer. If x is an integer, say m, then $x_o = 0$, x = [x] = m, and Eqs. (21) and (22) reproduce the known results [6] for the sum and alternate sum of m consecutive integers which are raised to the n^{th} power.

3. Three-term one-dimensional recurrence relations

A. Legendre polynomials

The Legendre polynomials $P_m(z)$ satisfy the three-term recurrence relation [6]:

$$P_m(z) = \frac{2m-1}{m} z\, P_{m-1}(z) - \frac{m-1}{m} P_{m-2}(z) \quad , \quad m \ge 1 \quad , \tag{23a}$$

with the initial values:

$$P_0(z) = 1 \quad , \quad P_{-1}(z) = 0 \quad . \tag{23b}$$

A slight generalization of this problem is obtained by adding an inhomogeneous term I(m; z) to the right-hand side of Eq. (23a) keeping the same initial values [5]:

$$B_m(z) = \frac{2m-1}{m} z\, B_{m-1}(z) - \frac{m-1}{m} B_{m-2}(z) + I(m;z) \quad , \quad m \ge 1 \quad . \tag{24}$$

Set A contains two displacement vectors in the direction of the positive m-axis and of magnitude 1 and 2, respectively. The functions associated with these displacements are:

$$f_1(m) = \frac{2m-1}{m} z \quad , \quad f_2(m) = \frac{m-1}{m} \quad . \tag{25}$$

In terms of the combinatorics functions associated with this problem, the solution is given as:

$$B_m(z) = C(0,m) + \sum_{j=1}^{m} I(j;z)\, C(j,m) \quad . \tag{26}$$

The combinatorics function C(0,m) is the Legendre polynomial $P_m(z)$. To complete the construction of the general solution, one has to compute C(j,m) for $m > j \geq 1$. The result is [5]:

$$C(j,m) = \sum_{p=0}^{[(m-j)/2]} (-1)^p\, z^{m-j-2p}\, \beta \quad , \tag{27a}$$

where [q] refers again to the integer part of q, and β depends on j, m and p according to:

$$\beta = 2^{m-j} \frac{j!\Gamma(m+\frac{1}{2})}{m!\Gamma(j+\frac{1}{2})} \sum_{k=0}^{p} \frac{(-1)^k}{2^p k!(p-k)!} \frac{m!}{(m-2p+2k)!} \frac{(2k+j-1)!}{(j-1)!}$$
$$\times \frac{\Gamma(m+\frac{1}{2}-p+k)\Gamma(k+j-\frac{1}{2})}{\Gamma(m+\frac{1}{2})\Gamma(2k+j-\frac{1}{2})} \quad . \tag{27b}$$

One may check that, for j = 0, there is only one non-zero term in the k-summation, the one for which k = 0. This, in turn, allows one to extend the validity of Eq. (27) to j = 0, and recover the expression of the Legendre polynomial as C(j,m), evaluated at j = 0.

B. Fibonacci-like recurrence relation

Consider the Fibonacci-like recurrence relation [7],

$$B_m = a\, B_{m-1} + b\, B_{m-p} \quad , \quad p \geq 2 \quad , \tag{28a}$$

subject to the initial values:

$$B_{\ell-p} = \lambda_\ell \quad , \quad \ell = 1, 2, ..., p - 1 \quad , \quad B_0 = \lambda_0 \quad . \tag{28b}$$

The Fibonacci numbers are the solutions of Eqs. (28) when a = b = 1, p = 2, λ_1 = 0 and λ_2 = 1. The standard method of solving Eqs. (28) would be to search for particular solutions of the form:

$$B_m = R^m \quad , \tag{29}$$

which in turn requires R to be a root of the characteristic equation

$$R^p - a\, R^{p-1} - b = 0 \quad . \tag{30}$$

Let R_k be one of the p roots with index k varying from 1 to p. Then the solution of Eq. (28a) is a linear combination of these p special solutions,

$$B_m = \sum_{k=1}^{p} L_k R_k^m \quad , \tag{31}$$

with the L_k's satisfying the initial value conditions (28b). The combinatorics function technique provides a simpler way of obtaining the analytical solution for any p ≥ 2. Here set A is made of two displacement vectors along the positive m-axis of magnitudes 1 and p, with the associated functions $f_1(m) = a$ and $f_p(m) = b$. Consider all paths reaching the evaluation point of coordinate m ≥ 1, and leaving any of the boundary points of coordinate – j (j = 0, 1, ... , p – 1) while avoiding all other boundary points. Thus, the first displacement δ_1 for all of these paths is restricted to be equal to p for as long as j ≠ 0, and the combinatorics function C(–j, m) follows as [7]:

$$C(-j, m) = \sum_{k=0}^{\left[\frac{m-j}{p}\right]} a^{m-j-kp}\, b^{k+1-\delta_{0j}} \binom{m-j-k(p-1)}{k} \quad , \tag{32}$$

where the quantity in brackets represents a binomial coefficient. The solution to Eq. (28a) satisfying the initial value conditions (28b) is:

$$B_m = \sum_{j=0}^{p} \lambda_j\, C(-j, m) \quad . \tag{33}$$

In this manner, we avoided calculating the roots of the characteristic equation and the p coefficients L_k in terms of the p initial values λ_j. On the other hand, the equivalence between the two methods yields a sum rule valid for $m \geq 0$, $p \geq 2$ and for any values of a and b, namely [7],

$$\sum_{k=0}^{\left[\frac{m}{p}\right]} a^{m-kp}\, b^k \binom{m-k(p-1)}{k} = \sum_{k=1}^{p} \frac{R_k^{m+1}}{a+bpR_k^{1-p}} \quad . \tag{34}$$

4. The Schrödinger equation with a power-type potential

Consider a particle of mass m in a central potential

$$V(r) = K\, r^N \quad , \tag{35}$$

where N is a positive integer. The radial part R(r) of the wave function describing the stationary state of the particle with energy E is written as

$$R(r) = U(r)/r \quad , \tag{36}$$

with U(r) the well-behaved solution of the radial Schrödinger equation:

$$\frac{d^2U_\ell}{d\rho^2} - \left(\frac{\ell(\ell+1)}{\rho^2} + \rho^N - t\right) U_\ell = 0 \quad . \tag{37}$$

Here, ℓ is the orbital quantum number, and t and ρ are dimensionless energy and radial variable parameters:

$$t = \left(\frac{\hbar}{2m}\right)^{\frac{N}{N+2}} K^{\frac{2}{N+2}}\, E \quad ; \quad \rho = \left(\frac{2mK}{\hbar^2}\right)^{\frac{1}{N+2}} r \quad . \tag{38}$$

The behavior of U_ℓ for large values of ρ is the same for all values of ℓ. This behavior is that of the solution of Eq. (37) with $\ell=0$, namely,

$$\frac{d^2U_o}{d\rho^2} - (\rho^N - t)\, U_o = 0 \quad . \tag{39}$$

For the linear potential (N = 1), the well-behaved solution of Eq. (39) is the Airy function [6], $Ai(\rho - t)$. Since R(r) must be a well-behaved function in the limit as r approaches zero, one should also require that $Ai(\rho - t) = 0$ as ρ approaches zero; thus:

$$Ai(-t) = 0 \quad . \tag{40}$$

The roots of this equation provide the s-state ($\ell = 0$) energy eigenvalues for the linear potential problem (N = 1). Here we intend to develop an energy eigenvalue equation valid for positive integer values of N which reduces to Eq. (40) when setting N = 1 and $\ell = 0$.

By factoring out the non-singular behavior of U_ℓ near the origin, we search for a series solution of the form:

$$U_\ell = \rho^{\ell+1} \sum_{m=0}^{\infty} b_m \rho^m \quad . \tag{41}$$

Substituting this series into Eq. (37) yields the three-term recurrence relation:

$$m(m + 2\ell + 1)\, b_m = -t\, b_{m-2} + b_{m-N-2}, \quad m > 0 \quad , \tag{42a}$$

subject to the initial value conditions:

$$b_{-m} = 0 \quad \text{for} \quad m > 0 \quad , \quad b_0 \neq 0 \quad . \tag{42b}$$

We intend to express b_m in terms of higher order terms with the boundary at points on the m-axis with coordinates $\{m + M + j \; ; \; j = 0, 1, ..., N + 1\}$, with M being at this point an unspecified positive integer. This is why we use an equivalent form of Eq. (42a),

$$b_m = (m + N + 2)(m + N + 2\ell + 3)\, b_{m+N+2} + t\, b_{m+N} \quad . \tag{43}$$

Set A for this recurrence relation consists of displacements in the direction of the negative m-axis with magnitudes N + 2 and N, and their associated functions are:

$$F_{-(N+2)}(m) = (m + N + 2)(m + N + 2\ell + 3) \quad , \quad f_{-N}(m) = t \quad . \tag{44}$$

With C(m + M + j, m) designating the combinatorics function, the solution of Eq. (43) in terms of the coefficients b_{m+M+j} (j = 0, 1, ..., N) is:

$$b_m = \sum_{j=0}^{N+1} b_{m+M+j}\, C(m+M+j, m) \quad . \tag{45}$$

This expression holds for any value of $m \geq 0$. Furthermore, Eq. (42a) with the boundary value conditions (42b) implies

$$2(\ell+1)\, b_1 = -\, t\, b_{-1} + b_{-N-1} = 0 \quad . \tag{46}$$

Combining Eqs. (45) and (46) yields

$$b_1 = \sum_{j=0}^{N+1} b_{1+M+j}\, C(1+M+j, 1) \quad . \tag{47}$$

Next, we consider a double series expansion of $U_o(\rho)$ of the form [9]

$$U_o(\rho) = \sum_{n=0}^{\infty} \rho^n \left\{ \sum_{i=0}^{\infty} (-t)^i A(n,i) \right\} \equiv \sum_{n=0}^{\infty} \rho^n b_n(\ell = 0, t) \quad , \tag{48}$$

which exists for $N \neq 2$, and its expansion coefficients A(n, i) satisfy the recurrence relation:

$$n(n-1)\, A(n, i+1) - A(n-2, i) - A(n-N-2, i+1) = 0 \quad , \tag{49a}$$

with the initial value conditions:

$$A(-n, i) = 0 \quad \text{for } n > 0 \quad , \quad A(0, 0) \neq 0 \quad . \tag{49b}$$

Since the asymptotic behavior of $U_\ell(\rho)$ is the same as that of $U_o(\rho)$ for all values of ℓ, in the limit as n becomes infinite, coefficient $b_n(\ell, t)$ should be proportional to $b_n(\ell = 0, t)$:

$$b_n(\ell, t) \propto b_n(\ell = 0, t) = \sum_{i=0}^{\infty} (-t)^i A(n, i) \quad . \tag{50}$$

Using Eq. (50) in taking the limit of Eq. (47) as M approaches infinity yields:

$$b_1 \propto H_{\ell N}(t) =$$

$$= \sum_{i=0}^{\infty} (-t)^i \left\{ \lim_{M\to\infty} \sum_{j=0}^{N+1} A(1+M+j,i)\, C(1+M+j,1) \right\} = 0 \quad . \tag{51}$$

This infinite order polynomial in t is the generalization of the one obtained in Ref. 10 for the linear potential (N = 1) and holds for all positive values of N except N = 2. Excluding the harmonic potential (N = 2), the roots of $H_{\ell N}(t)$ are the energy eigenvalues of the positive power potential problem. With an appropriate choice of A(0, 0), $H_{\ell N}(t)$ reduces to Ai(– t) for N = 1 and ℓ = 0. Unfortunately, we have been unable to find a closed form expression for the expansion coefficients for arbitrary N and ℓ. In the linear potential case (N = 1), we were able to compute in closed form the first few lower order terms for ℓ = 1.

5. Summary

Multidimensional linear recurrence relations can be formally solved in terms of the combinatorics functions. A combinatorics function depends on a *boundary* point and an *evaluation* point. Boundary points are determined by the initial value conditions and the evaluation points are those at which one would like to determine the value of the unknown in terms of the initial values. The construction of a given combinatorics function is based on all possible paths connecting a boundary point to an evaluation point. These paths are made of discrete displacements. The magnitudes and directions of these displacements are readily identified from the recurrence relation. The number of such displacements is one less than the number of terms in the recurrence relation, and a one-to-one correspondence is established between these displacements and the coefficients appearing in the recurrence relation. A path connecting a boundary point to an evaluation point is then made of a given sequence of displacements. With such a path one associates the product of the corresponding coefficients evaluated at the successive intermediate points encountered along the path. The construction of the combinatorics function for a given set of boundary and evaluation points follows as the sum of such products, corresponding to all possible paths leaving the boundary point and reaching the evaluation point, while avoiding all other boundary points. For a few cases we have shown how known results using other methods are recovered using the combinatorics function technique, while providing some natural generalizations. Not presented in this article is the use of the combinatorics function technique to include the solution of linearly coupled recurrence relations [11]. Such relations involve a set of multidimensional functions $B_i(\vec{M})$. The value of a given function B_j at the evaluation point M is linearly related not only to its values at other evaluation points but also to the values of the remaining functions evaluated at shifted arguments. A general method for decoupling these equations has been presented in Ref. 11. In the case of linearly coupled relations with constant coefficients [12], the decoupling is much simpler to achieve than in the general case. Applications of this decoupling has been extremely useful in the study of Ising models [13, 14]. This technique was also instrumental in developing a computational method for the exact study of low temperature adsorption patterns on crystal surfaces of finite width and infinite length [15]. We found that the crystallization patterns on semi-infinite surfaces without periodic boundaries have characteristics which fit exact analytic expressions as a function of the width of the surface.

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