Density matrix for photons in a cavity

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The transient behavior of the density operator for radiation in a single-mode cavity at a finite temperature is considered. Any initial state will evolve toward thermal equilibrium because of the interaction with the mirrors. This steady state is determined uniquely by the temperature, but the transient state depends on the initial conditions. The equation of motion for the matrix elements of the density operator is solved analytically, given an arbitrary initial state. The factorial moments, the generating function, and the time-dependent spectral distribution are also obtained. The results yield known expressions in the appropriate limits.

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1. INTRODUCTION

We consider radiation in a single-mode cavity at temperature T. If the radiation were in thermal equilibrium with the cavity mirrors, then the average number of photons would be[1]

$$n_{\mathrm{eq}} = \frac{1}{\exp(\hbar\omega_c/kT) - 1}, \tag{1.1}$$

where ω_c is the resonance frequency and k is Boltzmann's constant. Coupling between the radiation and the mirrors leads to relaxation toward thermal equilibrium, and in a finite-Q cavity the coupling parameter is $K = \omega_c/Q$. The equation of motion for the density operator ρ of the radiation is[2–4]

$$i\frac{\mathrm{d}\rho}{\mathrm{d}t} = (L_r - iL_c)\rho\,. \tag{1.2}$$

The Liouville operator L_r accounts for the free evolution and is given by

$$L_r\rho = \omega_c[a^\dagger a, \rho], \tag{1.3}$$

where $a^\dagger$ and a are the usual photon-creation and -annihilation operators, respectively. The mirrors of the cavity give rise to damping in the time evolution, and this is included in the equation of motion by means of the Liouvillian L_c. This operator is defined as

$$\begin{aligned} L_c\rho = {}& {}^1\!/\!_2 K n_{\mathrm{eq}}(aa^\dagger\rho + \rho aa^\dagger - 2a^\dagger\rho a) \\ & + {}^1\!/\!_2 K(n_{\mathrm{eq}} + 1)(a^\dagger a\rho + \rho a^\dagger a - 2a\rho a^\dagger)\,. \end{aligned} \tag{1.4}$$

We consider n_{eq} a free parameter of the system, representing the finite temperature. This n_{eq} is equal to the average number of photons in the steady state, $t \to \infty$. For $T = 0$ we have $n_{\mathrm{eq}} = 0$, for which the terms representing thermal excitation in Eq. (1.4) vanish.

The formal solution of Eq. (1.2) is

$$\rho(t) = \exp[-i(L_r - iL_c)t]\rho(0), \tag{1.5}$$

showing that an arbitrary initial state $\rho(0)$ determines the solution for all $t > 0$. Of practical interest are the matrix elements with respect to number states, $\rho(t)_{\mathrm{nm}} = \langle n|\rho(t)|m\rangle$, and they are determined, in principle, by the initial matrix elements $\rho(0)_{\mathrm{nm}}$. The set $\{\rho(t)_{\mathrm{nm}}\}$ depends in a linear way on the initial set $\{\rho(0)_{\mathrm{nm}}\}$, as follows from Eq. (1.5). If we can express the matrix elements $\{\rho(t)_{\mathrm{nm}}\}$ as linear combinations of $\{\rho(0)_{\mathrm{nm}}\}$, then the coefficients of the transformation matrix are the matrix elements of the evolution operator $\exp[-i(L_r - iL_c)t]$. The evaluation of this operator would also be relevant for the Jaynes–Cummings model with cavity damping,[5–14] in which it is sometimes used to transform the equation of motion to the dissipation picture. Recently Mufti *et al.*[15] proposed a solution of Eq. (1.2) in the form of the Ansatz

$$\rho(t) = \exp[\phi(t)]\exp[\alpha(t)a^\dagger]\exp[\chi(t)a^\dagger a]\exp[\alpha(t)^* a]\,. \tag{1.6}$$

This Ansatz leads to a set of nonlinear equations for the unknown functions $\phi(t)$, $\alpha(t)$, and $\chi(t)$, and the initial values have to be determined from the factorization of $\rho(0)$ in the same form as in Eq. (1.6). It is not obvious how that can be accomplished in general for an arbitrary $\rho(0)$. This solution still involves exponentials of operators, although in a much simpler form than in Eq. (1.5).

Taking the matrix elements of Eq. (1.2) with respect to number states and using Eqs. (1.3) and (1.4) give

$$\begin{aligned} \frac{\mathrm{d}\rho_{\mathrm{nm}}}{\mathrm{d}\tau} = {}& -i(n - m)Q\rho_{\mathrm{nm}} - {}^1\!/\!_2 n_{\mathrm{eq}} \\ & \times [(n + m + 2)\rho_{nm} - 2\sqrt{nm}\,\rho_{n-1,m-1}] \\ & - {}^1\!/\!_2(n_{\mathrm{eq}} + 1)\{(n + m)\rho_{nm} \\ & - 2[(n + 1)(m + 1)]^{1/2}\rho_{n+1,m+1}\}, \\ n, m = {}& 0, 1, 2, \ldots, \end{aligned} \tag{1.7}$$

where we have set $\tau = Kt$. Of particular interest are the populations $p_n(\tau) = \langle n|\rho(\tau)|n\rangle$, $n = 0, 1, 2, \ldots$, which equal the probabilities of finding n photons in the cavity at time τ. For $m = n$, Eq. (1.7) simplifies to

$$\begin{aligned} \frac{\mathrm{d}p_n}{\mathrm{d}\tau} = {}& -n_{\mathrm{eq}}\{(n + 1)p_n - np_{n-1}\} \\ & - (n_{\mathrm{eq}} + 1)\{np_n - (n + 1)p_{n+1}\}, \end{aligned} \tag{1.8}$$

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which is a rate equation for the populations $p_n(\tau)$ of level $|n\rangle$. The first two terms are excitations from and to level $|n\rangle$, and the next two terms are decays to and from level $|n\rangle$. Solutions of an equation of the type of Eq. (1.7) have been found by an expansion in eigenfunctions and eigenvalues of the coefficient matrix defined by the right-hand side of Eq. (1.7).[16] Equation (1.8) has been solved for a variety of initial conditions by a generating-function technique.[17–20] If the density operator has a P representation, then Eq. (1.8) can be transformed into a Fokker–Planck equation for this P representation, which can subsequently be solved.[1,21] In this paper Eq. (1.7) is solved directly in terms of the initial matrix elements, and the result is applied to the evaluation of some quantities of interest.

2. GENERATING FUNCTION

Equation (1.7) couples only matrix elements with the same value of $m - n$. Therefore we set $m = n + l$, with $l = 0, 1, 2, \ldots$, and consider l fixed. Then Eq. (1.7) couples $\rho_{n,n+l}$ for $n = 0, 1, 2 \ldots$. Equation (1.7) can be simplified with the following transformation:

$$\gamma_n(\tau) = \exp[(n_{\text{eq}} + 1/2 - iQ)l\tau]\left[\frac{(n+l)!}{n!}\right]^{1/2}\rho_{n,n+l}(\tau), \tag{2.1}$$

which gives

$$\begin{aligned}\frac{\mathrm{d}\gamma_n}{\mathrm{d}\tau} = &-n_{\text{eq}}\{(n+1)\gamma_n - (n+l)\gamma_{n-1}\} \\ &-(n_{\text{eq}}+1)\{n\gamma_n - (n+1)\gamma_{n+1}\},\end{aligned} \tag{2.2}$$

and we set $\gamma_{-1} \equiv 0$. Equation (2.2) for $\gamma_n(\tau)$ is almost identical to Eq. (1.8) for $p_n(\tau)$. Equations of this type are most conveniently solved through the use of a generating function.[22–24] Let

$$g(x, \tau) = \sum_{n=0}^{\infty} x^n \gamma_n(\tau), \tag{2.3}$$

with x an auxiliary parameter. Then we multiply Eq. (2.2) by x^n and sum over n. The resulting equation can then be written as

$$\frac{\partial g}{\partial \tau} = (1-x)[1 + n_{\text{eq}}(1-x)]\frac{\partial g}{\partial x} - n_{\text{eq}}[1 - x(l+1)]g, \tag{2.4}$$

a partial differential equation for $g(x, \tau)$.

Equation (2.4) can be solved with Laplace transform in τ (see Appendix A) or with the method of characteristics.[25,26] The solution is

$$g(x, \tau) = \frac{\exp(n_{\text{eq}}l\tau)}{[1 + u(1-x)]^{l+1}}\, g(\xi, 0), \tag{2.5}$$

where we replace x in the initial generating function $g(x, 0)$ by the parameter

$$\xi = \frac{1 + v(1-x)}{1 + u(1-x)}. \tag{2.6}$$

Here we have introduced the abbreviations

$$u = n_{\text{eq}}[1 - \exp(-\tau)], \tag{2.7}$$

$$v = n_{\text{eq}} - (n_{\text{eq}} + 1)\exp(-\tau). \tag{2.8}$$

3. MATRIX ELEMENTS

The x dependence of the generating function determines the quantities $\gamma_n(\tau)$, because they are the Taylor coefficients in a series expansion around $x = 0$, according to Eq. (2.3). Hence these quantities can be determined by n-fold differentiation as

$$\gamma_n(\tau) = \frac{1}{n!}\frac{\partial^n}{\partial x^n} g(x, \tau)\bigg|_{x=0}. \tag{3.1}$$

A complication here is that Eq. (2.5) contains the initial generating function, with ξ as a variable. To express $\gamma_n(\tau)$ in terms of a linear combination of all $\gamma_n(0)$ we set

$$g(\xi, 0) = \sum_{k=0}^{\infty} \xi^k \gamma_k(0) \tag{3.2}$$

in Eq. (2.5) and then substitute expression (2.6) for ξ. Carrying out the n-fold differentiation then yields

$$\begin{aligned}\gamma_n(\tau) = &\exp(n_{\text{eq}}l\tau)\left(\frac{u}{1+u}\right)^n \sum_{k=0}^{\infty} \gamma_k(0)\frac{k!}{(k+l)!}\frac{(1+v)^k}{(1+u)^{k+l+1}} \\ &\times \sum_m \frac{(k+l+n-m)!}{(n-m)!m!(k-m)!}\left[-\frac{v(1+u)}{u(1+v)}\right]^m.\end{aligned} \tag{3.3}$$

The summation over m effectively runs up to $m = \min(n, k)$ because of the factorials in the denominator. Therefore the series over m is a finite sum.

The matrix elements of the density operator now follow from transformation (2.1). We find that

$$\begin{aligned}\rho_{n,n+l}(\tau) = &\exp[(iQ - 1/2)l\tau]\left[\frac{n!}{(n+l)!}\right]^{1/2}\left(\frac{u}{1+u}\right)^n \\ &\times \sum_{k=0}^{\infty} \rho_{k,k+l}(0)\left[\frac{k!}{(k+l)!}\right]^{1/2}\frac{(1+v)^k}{(1+u)^{k+l+1}} \\ &\times \sum_m \frac{(k+l+n-m)!}{(n-m)!m!(k-m)!}\left[-\frac{v(1+u)}{u(1+v)}\right]^m\end{aligned} \tag{3.4}$$

for $n, l = 0, 1, 2, \ldots$. With $Q\tau = \omega_c t$ we see that $\rho_{n,n+l}(\tau)$ oscillates with the Bohr frequency $\omega_c l$, as it should. The remaining matrix elements follow from

$$\rho_{n+l,n}(\tau) = \rho_{n,n+l}(\tau)^*, \tag{3.5}$$

because ρ is Hermitian.

For $l = 0$ the coherences go over into the probabilities $p_n(\tau)$. When we write

$$p_n(\tau) = \sum_{m=0}^{\infty} X_{n,m}(\tau)p_m(0), \tag{3.6}$$

then $X_{n,m}(\tau)$ is given by

$$\begin{aligned}X_{n,m}(\tau) = &\frac{u^n}{(1+u)^{n+1}}\left(\frac{1+v}{1+u}\right)^m \sum_j \frac{(m+n-j)!}{(n-j)!j!(m-j)!} \\ &\times \left[-\frac{v(1+u)}{u(1+v)}\right]^j.\end{aligned} \tag{3.7}$$

The summation over j runs from $j = 0$ to $j = \min(n, m)$. The quantity $X_{n,m}(\tau)$ can be considered the Green's function, as it equals the probability $p_n(\tau)$ when the initial state is the number state $|m\rangle\langle m|$. In terms of the hypergeometric function $F(a, b, c; z)$,[27] $X_{n,m}(\tau)$ can be expressed as

$$X_{n,m}(\tau) = \frac{u^n}{(1+u)^{n+1}}\left(\frac{1+v}{1+u}\right)^m \binom{n+m}{n} \times F\left[-n, -m, -n-m; \frac{v(1+u)}{u(1+v)}\right], \tag{3.8}$$

and with the identity

$$\binom{n+m}{m} F(-n, -m, -n-m; z) = F(-n, -m, 1; 1-z) \tag{3.9}$$

this can be simplified to

$$X_{n,m}(\tau) = \frac{u^n}{(1+u)^{n+1}}\left(\frac{1+v}{1+u}\right)^m F\left[-n, -m, 1; \frac{u-v}{u(1+v)}\right]. \tag{3.10}$$

4. FACTORIAL MOMENTS

The factorial moment $s_k(\tau)$ of the probability distribution is defined as the average of $n!/(n-k)!$ for $k = 0, 1, 2, \ldots,$ e.g.,

$$s_k(\tau) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\, p_n(\tau). \tag{4.1}$$

The inverse of this relation is

$$p_n(\tau) = \sum_{k=0}^{\infty} \frac{(-1)^k}{n!k!}\, s_{n+k}(\tau), \tag{4.2}$$

showing that either the set $\{p_n(\tau)\}$ or the set $\{s_k(\tau)\}$ uniquely determines the probability distribution. Moreover, both sets are related through the generating function according to

$$g(x, \tau) = \sum_{n=0}^{\infty} x^n p_n(\tau) = \sum_{k=0}^{\infty} \frac{(x-1)^k}{k!}\, s_k(\tau), \tag{4.3}$$

showing that $\{p_n(\tau)\}$ are the Taylor coefficients for an expansion around $x = 0$ and $\{s_k(\tau)\}$ are the Taylor coefficients for an expansion around $x = 1$ of the same function.

The evaluation of the factorial moments proceeds in the same way as the derivation of the matrix elements from the generating function. From Eq. (4.3) it follows that

$$s_k(\tau) = \left.\frac{\partial^k}{\partial x^k}\, g(x, \tau)\right|_{x=1}. \tag{4.4}$$

We set $l = 0$ in Eq. (2.5) and replace the initial generating function by

$$g(\xi, 0) = \sum_{l=0}^{\infty} \frac{(\xi-1)^l}{l!}\, s_l(0), \tag{4.5}$$

with ξ given by Eq. (2.6). Carrying out the k-fold differentiation and letting $x \to 1$ yield

$$s_k(\tau) = \sum_{l=0}^{k} \left(\frac{k!}{l!}\right)^2 \frac{1}{(k-l)!}\, u^{k-l} \exp(-l\tau) s_l(0). \tag{4.6}$$

The factorial moment $s_k(\tau)$ is determined by the initial factorial moments $s_l(0)$ for $l \le k$ only, whereas the probability $p_n(\tau)$ depends on all initial probabilities.

For $k = 0$ we have

$$s_o(\tau) = s_o(0) = \sum_{n=0}^{\infty} p_n(\tau) = \mathrm{Tr}\ \rho(\tau) = 1. \tag{4.7}$$

The significance of the higher factorial moments lies in the fact that these are averages over the probability distribution and therefore relate to observable quantities. For $k = 1$ we obtain the average number of photons in the cavity at time τ. We find that

$$\begin{aligned}\overline{n}(\tau) = s_1(\tau) &= u + s_1(0)\exp(-\tau) \\ &= n_{\mathrm{eq}} + [\overline{n}(0) - n_{\mathrm{eq}}]\exp(-\tau),\end{aligned} \tag{4.8}$$

showing that $\overline{n}(\infty) = n_{\mathrm{eq}}$, as it should. The variance in the photon distribution is related to the second factorial moment according to

$$\mathrm{Var}(\tau) = \sum_{n=0}^{\infty} [n - \overline{n}(\tau)]^2 p_n(\tau) = s_2(\tau) + s_1(\tau) - s_1(\tau)^2, \tag{4.9}$$

and with Eq. (4.6) this becomes

$$\begin{aligned}\mathrm{Var}(\tau) &= u(u+1) + (2u+1)\overline{n}(0)\exp(-\tau) \\ &\quad + [\mathrm{Var}(0) - \overline{n}(0)]\exp(-2\tau)\end{aligned} \tag{4.10}$$

in terms of the parameter u. In the steady state we have $u = n_{\mathrm{eq}}$, and therefore $\mathrm{Var}(\infty) = n_{\mathrm{eq}}(n_{\mathrm{eq}} + 1)$, as expected for a thermal distribution. The factorial moments for $k = 0, 1, 2$ can also be derived directly from Eq. (1.8). If we multiply Eq. (1.8) by 1, n, and $n(n-1)$, and then sum over n, we obtain the relations

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \sum_{n=0}^{\infty} p_n(\tau) = 0, \tag{4.11}$$

$$\frac{\mathrm{d}\overline{n}}{\mathrm{d}\tau} = n_{\mathrm{eq}} - \overline{n}, \tag{4.12}$$

$$\frac{\mathrm{d}s_2}{\mathrm{d}\tau} = -2s_2 + 4n_{\mathrm{eq}}\overline{n}, \tag{4.13}$$

with the solutions given above. This procedure becomes increasingly more complicated for the higher factorial moments.

5. SPECIAL CASES

A. Zero Temperature

In the limit of zero temperature we have $n_{\mathrm{eq}} = 0$, $u = 0$, and $v = -\exp(-\tau)$. The factorial moments simplify to[28,29]

$$s_k(\tau) = \exp(-k\tau) s_k(0), \tag{5.1}$$

and the quantities $X_{n,m}(\tau)$ become

$$X_{n,m}(\tau) = \begin{cases} 0 & m < n \\ \binom{m}{n} \exp(-n\tau)[1 - \exp(-\tau)]^{m-n} & m \ge n \end{cases}, \tag{5.2}$$

as follows from Eq. (3.7). This gives for the probabilities

$$p_n(\tau) = \sum_{m=n}^{\infty} \binom{m}{n} \exp(-n\tau)[1 - \exp(-\tau)]^{m-n} p_m(0). \tag{5.3}$$

This solution has been known for a long time[30] and has been derived in many different ways. It is a Bernoulli distribution over the initial distribution; $\exp(-\tau)$ is the probability that a photon is still in the cavity at time τ when it was in the cavity at time $\tau = 0$.

B. Steady State

For $\tau \to \infty$ we have from Eq. (3.4)

$$\rho_{n,n+l}(\infty) = \delta_{l,0} p_n(\infty), \tag{5.4}$$

e.g., the coherences die out. For $\tau \to \infty$ we have $u = v = n_{\mathrm{eq}}$, and with Eq. (3.7) this gives

$$X_{n,m}(\tau) = \frac{n_{\mathrm{eq}}{}^n}{(1 + n_{\mathrm{eq}})^{n+1}} \sum_j \frac{(m + n - j)!}{(n - j)!j!(m - j)!}(-1)^j. \tag{5.5}$$

The summation over j can be performed:

$$\sum_j \frac{(m + n - j)!}{(n - j)!j!(m - j)!}(-1)^j = 1\,, \tag{5.6}$$

which becomes independent of m. We can prove relation (5.6) by multiplying both sides by z^n and summing over n. Both sides then yield the same sum, which proves Eq. (5.6). With Eq. (3.6) we obtain

$$p_n(\infty) = \frac{n_{\mathrm{eq}}{}^n}{(1 + n_{\mathrm{eq}})^{n+1}}\,, \tag{5.7}$$

the thermal distribution. The corresponding factorial moments are

$$s_k(\infty) = k! n_{\mathrm{eq}}{}^k\,, \tag{5.8}$$

as follows from Eq. (4.6) (only the $l = 0$ term survives).

C. Thermal State

If the initial state is a thermal state with n_o photons, then the initial distribution is given by Eq. (5.7) with n_{eq} replaced by n_o. This corresponds, for instance, to a sudden change in temperature of the cavity. The initial factorial moments are $s_l(0) = l! n_o{}^l$. When we substitute this into Eq. (4.6) the summation can be performed, with the result that

$$s_k(\tau) = k! \overline{n}(\tau)^k, \tag{5.9}$$

where $\overline{n}(\tau) = n_{\mathrm{eq}} + (n_o - n_{\mathrm{eq}})\exp(-\tau)$. This shows that the distribution is a thermal distribution at all times. The probabilities are given by Eq. (5.7) with n_{eq} replaced by $\overline{n}(\tau)$.

D. Coherent State

Let the initial state be a coherent state, e.g., $\rho(0) = |\alpha\rangle\langle\alpha|$, with α complex. The initial matrix elements are then

$$\rho_{k,k+l}(0) = \frac{|a|^{2k}(\alpha^*)^l}{[k!(k + l)!]^{1/2}} \exp(-|\alpha|^2)\,. \tag{5.10}$$

Substitution into the general solution [Eq. (3.4)] leads to a series of the type

$$\sum_{k=m}^{\infty} \frac{(k + l + n - m)!}{(k + l)!(k - m)!} z^k = (n - m)! z^m e^z L_{n-m}^{(l+m)}(-z)\,, \tag{5.11}$$

with $L_n^{(k)}(x)$ a generalized Laguerre polynomial. The summation over m in Eq. (3.4) then has the form

$$\sum_{m=0}^{n} \frac{(-x)^m}{m!} L_{n-m}^{(l+m)}(-y) = L_n^{(l)}(x - y)\,. \tag{5.12}$$

We can prove Eqs. (5.11) and (5.12) by showing that both sides have the same generating function. Combining everything then yields for the matrix elements of the density operator

$$\rho_{n,n+l}(\tau) = \exp\left[-\frac{|\alpha|^2\exp(-\tau)}{1 + u} + (iQ - 1/2)l\tau\right] \times \left[\frac{n!}{(n + l)!}\right]^{1/2} \frac{u^n(\alpha^*)^l}{(1 + u)^{n+l+1}} L_n^{(l)}\left[-|\alpha|^2 \frac{\exp(-\tau)}{u(1 + u)}\right]. \tag{5.13}$$

When we set $l = 0$, these are probabilities $p_n(\tau)$, and we recognize this as a noncentral negative binomial distribution. Such states were introduced in quantum optics as a superposition of a coherent state and a thermal state, and they form an interpolation between these two states. The numbers of coherent and thermal photons are

$$n_c = |\alpha|^2 \exp(-\tau), \qquad n_t = u = n_{\mathrm{eq}}[1 - \exp(-\tau)], \tag{5.14}$$

respectively. The properties of these states have been studied extensively.[31–36] Such states are also the output of a linear amplifier with a coherent input,[37,38] which provides a different way of generating these states. The initial factorial moments are $s_l(0) = |\alpha|^{2l}$, and with Eq. (4.6) they give

$$s_k(\tau) = k! u^k L_k\left\{-\frac{|\alpha|^2}{n_{\mathrm{eq}}[\exp(\tau) - 1]}\right\}. \tag{5.15}$$

6. CORRELATION FUNCTION

The spectral distribution of the radiation in the cavity is determined by the correlation function $\langle a^\dagger(t_1)a(t_2)\rangle$, with $t_2 \geq t_1 \geq 0$. In the Schrödinger picture this is

$$\langle a^\dagger(t_1)a(t_2)\rangle = \mathrm{Tr}\ a \exp[-i(L_r - iL_c)(t_2 - t_1)][\rho(t_1)a^\dagger] \tag{6.1}$$

in terms of the evolution operator for the density matrix, as shown in Eq. (1.5). Taking the trace gives

$$\langle a^\dagger(t_1)a(t_2)\rangle = \sum_{n=0}^{\infty} \sqrt{n + 1}\,\langle n + 1|\{\exp[-i(L_r - iL_c)(t_2 - t_1)] \times [\rho(t_1)a^\dagger]\}|n\rangle\,. \tag{6.2}$$

The $(n + 1, n)$ matrix element is given by the complex conjugate of Eq. (3.4), with $l = 1$. The initial values now become $\rho_{k+1,k}(0) \to \langle k + 1|[\rho(t_1)a^\dagger]|k\rangle = \sqrt{k + 1}\,\rho(t_1)_{k+1,k+1}$, and this gives

$$\langle a^\dagger(t_1)a(t_2)\rangle = \exp[-(iQ + 1/2)\tau] \times \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \rho(t_1)_{k+1,k+1} \frac{u^n}{(1 + u)^{n+2}}\left(\frac{1 + v}{1 + u}\right)^k \times \sum_m \frac{(k + n + 1 - m)!}{(n - m)!m!(k - m)!}\left[-\frac{v(1 + u)}{u(1 + v)}\right]^m, \tag{6.3}$$

with $\tau = K(t_2 - t_1)$. First we perform the summation over n, and then the summation over m. Then Eq. (6.3) simplifies to

$$\langle a^\dagger(t_1)a(t_2)\rangle = \exp[-(iQ + 1/2)\tau]\sum_{k=0}^{\infty}(k+1)\rho(t_1)_{k+1,k+1}\,, \tag{6.4}$$

and this is

$$\langle a^\dagger(t_1)a(t_2)\rangle = \exp[-(i\omega_c + 1/2K)(t_2 - t_1)]\overline{n}(t_1)\,, \tag{6.5}$$

with the average number of photons given by Eq. (4.8) with $\tau \rightarrow Kt_1$. It appears that the temperature dependence of the correlation function enters only through $\overline{n}(t_1)$. The regression $t_1 \rightarrow t_2$ is temperature independent. Also, the only dependence on the initial density operator is through $\overline{n}(t_1)$, which is determined by $\overline{n}(0)$ only.

7. SPECTRUM

The spectral distribution of the radiation in the cavity is essentially time dependent, because the density operator evolves in time. Suppose that the radiation is prepared in a state $\rho(0)$ at time $t = 0$ and that we start the frequency measurement at time $t = 0$. The electric field in the cavity is measured with a frequency filter with setting frequency ω. The time-dependent spectrum is then the photon-counting rate by a detector, after filtering. This is the physical spectrum of light,[39] and we shall indicate this spectrum by $I(\omega, t)$. It can be shown[40] that the physical spectrum can be expressed in terms of a quasi-spectrum $J(\omega, t)$ according to

$$I(\omega, t) = \int_0^t \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}\omega' s(\omega', t')J(\omega - \omega', t - t')\,, \tag{7.1}$$

where the smoothing function $s(\omega, t)$ depends only on the detector properties. The quasi-spectrum was introduced by Page[41] and by Lampard[42] and is given by

$$J(\omega, t) = \frac{\zeta}{\pi}\,\mathrm{Re}\int_0^t \mathrm{d}t' \exp[i\omega(t - t')]\langle a^\dagger(t')a(t)\rangle\,. \tag{7.2}$$

Here it is assumed that the positive frequency part of the electric field (in the Heisenberg picture) is proportional to the annihilation operator $a(t)$ and that overall constants are collected in the parameter ζ. If the correlation function is stationary, then $J(\omega, t)$ reduces to the Wiener–Khintchine spectrum for $t \rightarrow \infty$. When we take for the frequency filter an exponentially decaying function, then $s(\omega, t)$ becomes

$$s(\omega, t) = \frac{2\gamma^3}{\gamma^2 + \omega^2}\exp(-2\gamma t)\,. \tag{7.3}$$

This shows that the frequency resolution of the filter is γ, whereas the time resolution in the detection of the arrival of a photon is $1/\gamma$.

The correlation function in Eq. (7.2) was evaluated in Section 5, and the integral over t' is easily calculated. We obtain

$$\begin{aligned} J(\omega, t) = {} & \frac{\zeta}{\pi}\, n_{\mathrm{eq}}\,\mathrm{Re}\,\frac{\exp[(i(\omega - \omega_c) - 1/2K)t] - 1}{i(\omega - \omega_c) - 1/2K} \\ & + \frac{\zeta}{\pi}[\overline{n}(0) - n_{\mathrm{eq}}]\exp(-Kt) \\ & \times \mathrm{Re}\,\frac{\exp[(i(\omega - \omega_c) - 1/2K)t] - 1}{i(\omega - \omega_c) + 1/2K}\,. \end{aligned} \tag{7.4}$$

Then we introduce the dimensionless variables

$$\tau = Kt, \qquad \lambda = (\omega - \omega_c)/K, \qquad N = n_{\mathrm{eq}}/\overline{n}(0)\,, \tag{7.5}$$

work out the real parts, and suppress a factor of $\zeta\overline{n}(0)/\pi K$. This yields for the quasi-spectrum

$$\begin{aligned} J(\lambda, \tau) = {} & \frac{2}{1 + 4\lambda^2}\{\exp(-1/2\tau) \\ & \times [\cos(\lambda\tau) + 2\lambda\sin(\lambda\tau) - \exp(-1/2\tau)] \\ & + N[1 + \exp(-\tau) - 2\cos(\lambda\tau)\exp(-1/2\tau)]\}\,. \end{aligned} \tag{7.6}$$

The first part is independent of the temperature, and it vanishes in the long-time limit. The second part, proportional to n_{eq}, survives, and in the steady state it becomes

$$J(\lambda, \infty) = \frac{2N}{1 + 4\lambda^2}\,, \tag{7.7}$$

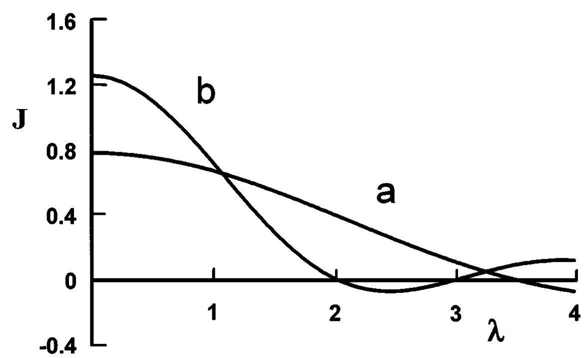

Fig. 1. Plot of the quasi-spectrum $J(\lambda, \tau)$ as a function of λ for $N = 1$. Curves a and b correspond to $\tau = 1$ and $\tau = 2$, respectively. The quasi-spectrum narrows with increasing τ and eventually approaches a Lorentzian with a HWHM of 1/2. Notice that the quasi-spectrum can become negative, indicating that it is not the observable spectral distribution of the radiation.

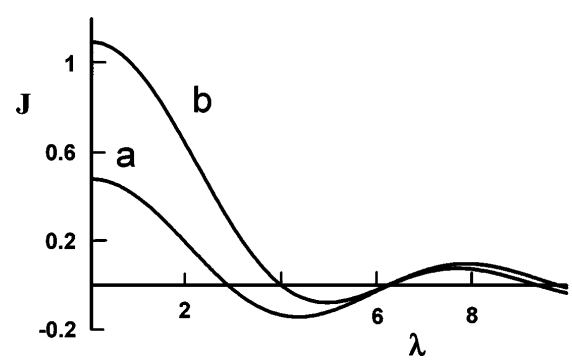

Fig. 2. Temperature dependence of the quasi-spectrum. Curve a represents zero temperature for the steady state ($N = 0$), so the cavity is cooling off. For curve b we took $N = 2$, giving $n_{\mathrm{eq}} = 2\overline{n}(0)$, and therefore this spectrum represents radiation that is warming up. The evolution time was taken as $\tau = 1$.

a Lorentzian with a half-width at half-maximum (HWHM) equal to 1/2 (and this is $K/2$ as a function of ω). The quasi-spectrum is symmetric around $\lambda = 0$. Figure 1 illustrates the behavior of $J(\lambda, \tau)$ for two values of τ, and Fig. 2 shows the effect of a finite temperature.

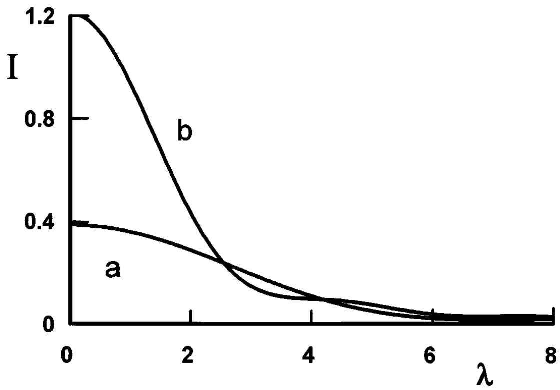

Fig. 3. Graphs of the physical spectrum $I(\lambda, \tau)$ as a function of λ for $\hat{\gamma} = 0.1$ and $N = 1$ [constant temperature between time zero and time τ, because $\overline{n}(\tau) = n_{\mathrm{eq}}$ according to Eq. (4.8)]. Curves a and b represent $\tau = 1$ and $\tau = 2$, respectively, and it is seen that the spectrum narrows with increasing time. The physical spectrum is positive by construction.

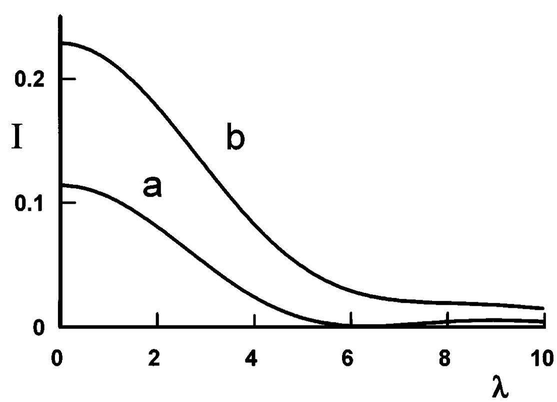

Fig. 4. Illustration of the temperature dependence of $I(\lambda, \tau)$. For curves a and b we have $N = 0$ and $N = 2$, respectively, and we took $\tau = \hat{\gamma} = 1$.

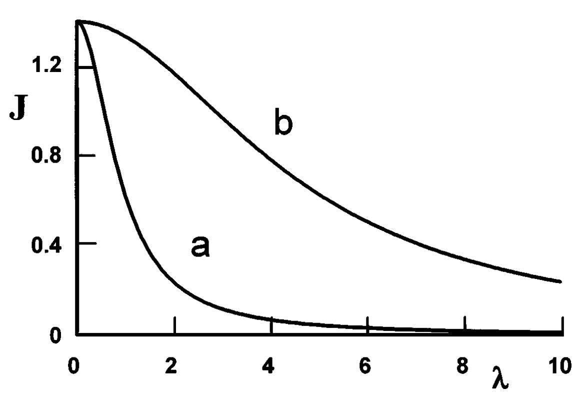

Fig. 5. Physical spectrum for $\tau = 20$ and $N = 1$. For curves a and b we have $\hat{\gamma} = 0.4$ and $\hat{\gamma} = 4$, respectively. The spectrum broadens significantly for increasing $\hat{\gamma}$, and for $\hat{\gamma} \gg 1$ the width is dominated by the resolution of the detector. Curve b has been multiplied by a factor of 50, so both curves have the same value at $\lambda = 0$.

The physical spectrum then follows from Eq. (7.1). The ω' integrals can be carried out by contour integration. We obtain

$$\begin{aligned} I(\omega, t) = 2\zeta\gamma^2 \Bigg(& n_{\mathrm{eq}} \operatorname{Re} \frac{1}{1/2K + \gamma - i(\omega - \omega_c)} \left\{ \frac{1 - \exp(-2\gamma t)}{2\gamma} \right. \\ & \left. - \frac{\exp[(i(\omega - \omega_c) - \gamma - 1/2K)t] - \exp(-2\gamma t)}{\gamma - 1/2K + i(\omega - \omega_c)} \right\} \\ & + [\overline{n}(0) - n_{\mathrm{eq}}] \operatorname{Re} \frac{1}{1/2K - \gamma + i(\omega - \omega_c)} \\ & \times \left\{ \frac{\exp[(i(\omega - \omega_c) - \gamma - 1/2K)t] - \exp(-2\gamma t)}{\gamma - 1/2K + i(\omega - \omega_c)} \right. \\ & \left. - \frac{\exp(-Kt) - \exp(-2\gamma t)}{2\gamma - K} \right\} \Bigg). \end{aligned} \tag{7.8}$$

Changing again to dimensionless parameters, introducing $\hat{\gamma} = \gamma/K$, working out the real part, and suppressing a factor of $2\zeta\gamma^2\overline{n}(0)/K^2$ then give for the physical spectrum

$$\begin{aligned} I(\lambda, \tau) = {} & N \frac{1 - \exp(-2\hat{\gamma}\tau)}{2\hat{\gamma}} \frac{\hat{\gamma} + 1/2}{(\hat{\gamma} + 1/2)^2 + \lambda^2} \\ & - \frac{N}{[(\hat{\gamma} + 1/2)^2 + \lambda^2][(\hat{\gamma} - 1/2)^2 + \lambda^2]} \\ & \times ((\hat{\gamma}^2 - 1/4 + \lambda^2)\{\cos(\lambda\tau)\exp[-(\hat{\gamma} + 1/2)\tau] \\ & - \exp(-2\hat{\gamma}\tau)\} + \lambda \sin(\lambda\tau)\exp[-(\hat{\gamma} + 1/2)\tau]) \\ & + 1/2 \frac{1 - N}{(\hat{\gamma} - 1/2)^2 + \lambda^2} \{\exp(-2\hat{\gamma}\tau) + \exp(-\tau) \\ & - 2\cos(\lambda\tau)\exp[-(\hat{\gamma} + 1/2)\tau]\}. \end{aligned} \tag{7.9}$$

This spectrum is also symmetric around $\lambda = 0$, and its steady-state value is

$$I(\lambda, \infty) = \frac{\hat{\gamma} + 1/2}{2\hat{\gamma}} \frac{N}{(\hat{\gamma} + 1/2)^2 + \lambda^2}, \tag{7.10}$$

a Lorentzian with HWHM $= \hat{\gamma} + 1/2$ around $\lambda = 0$ or $\gamma + 1/2K$ around $\omega = \omega_c$. Because of the finite observation time τ and the time evolution of the system, the spectrum deviates considerably from a Lorentzian for $\tau < \infty$. Typical behavior of $I(\lambda, \tau)$ is shown in Fig. 3 for two values of τ, and Fig. 4 illustrates the dependence on the temperature. The frequency resolution $\hat{\gamma}$ also affects the physical spectrum, as shown in Fig. 5. The spectrum broadens considerably with increasing $\hat{\gamma}$, as could be expected, and the peak value diminishes because the available energy is distributed over a larger frequency range.

8. CONCLUSIONS

The equation of motion for the density matrix of radiation in a single-mode cavity at finite temperature has been solved for its matrix elements, with Eq. (3.4) as the result. The populations at time τ (or t) are linear combinations of the populations at time zero, and this relation could be expressed in terms of hypergeometric functions, as shown in Eq. (3.10). A particular simple relation exists between the factorial moments at time τ and at time zero, as given by Eq. (4.6). In Section 5 it was shown that our solution reduces to known results in limiting cases. From the time evolution operator for the density operator we can obtain the time regression of correlation functions of field operators. In Section 6 the two-time field correlation was calculated, and this was subsequently used to evaluate the quasi-spectrum and the physical spectrum of the radiation in the cavity. It appears that both spectra depend strongly on the observation time, the spectral resolution of the detector, the cavity damping rate, and the temperature difference between time τ and time zero. Only in the long-time limit (thermal equilibrium) and for perfect frequency resolution by the detector do these spectra become Lorentzians with a HWHM equal to the cavity damping rate $K/2$.

APPENDIX A

To solve Eq. (2.4) we adopt a Laplace transform in τ. With

$$\tilde{g}(x,\, s) = \int_0^\infty \mathrm{d}\tau \, \exp(-s\tau) g(x,\, \tau) \tag{A1}$$

Eq. (2.4) becomes

$$\{s + n_{\text{eq}}[1 - x(l+1)]\}\tilde{g} + (x-1) \times [1 + n_{eq}(1-x)]\frac{\partial \tilde{g}}{\partial x} = g(x,\, 0)\,, \tag{A2}$$

where $g(x, 0)$ is assumed to be known. The generating function is defined as a Taylor series around $x = 0$. Equation (A2) is a linear first-order equation in x, and the general solution therefore contains one integration constant. For $x = 0$ we have $g(0, \tau) = \gamma_0(\tau)$, according to Eq. (2.3), and this is an unknown quantity. However, if we set $x = 1$ in Eq. (2.4) we get

$$\frac{\partial g}{\partial \tau} = n_{\text{eq}} l g\,, \tag{A3}$$

with the solution

$$g(1,\, \tau) = \exp(n_{\text{eq}} l \tau) g(1,\, 0) \tag{A4}$$

and the Laplace transform

$$\tilde{g}(1,\, s) = \frac{1}{s - n_{\text{eq}} l}\, g(1,\, 0)\,. \tag{A5}$$

This gives us the integration constant for Eq. (A2). The solution of Eq. (A2) is then

$$\tilde{g}(x,\, s) = \frac{[1 + n_{\text{eq}}(1-x)]^{s-1-l(n_{\text{eq}}+1)}}{(x-1)^{s - l n_{\text{eq}}}} \times \int_1^x \mathrm{d}\xi \, \frac{(\xi - 1)^{s-1-l n_{\text{eq}}}}{[1 + n_{\text{eq}}(1-\xi)]^{s - l(n_{\text{eq}}+1)}}\, g(\xi,\, 0)\,. \tag{A6}$$

That this solution reduces to Eq. (A5) in the limit $x \to 1$ follows from a Taylor expansion of the integrand around $\xi = 1$ and a term-by-term integration.

We can find the Laplace inverse of Eq. (A6) by making a change of integration variable $\xi \to \tau$ according to

$$\exp(-\tau) = \frac{1 + n_{\text{eq}}(1-x)}{1-x}\, \frac{1-\xi}{1 + n_{\text{eq}}(1-\xi)}\,. \tag{A7}$$

This yields the representation

$$\tilde{g}(x,\, s) = \int_0^\infty \mathrm{d}\tau \, \exp(-s\tau) \left[\frac{1 + n_{\text{eq}}(1-\xi)}{1 + n_{\text{eq}}(1-x)}\right]^{1+l(n_{\text{eq}}+1)} \times \left(\frac{1-x}{1-\xi}\right)^{l n_{\text{eq}}} g(\xi,\, 0)\,. \tag{A8}$$

But this identical in form to Eq. (A1), and therefore the inverse is

$$g(x,\, \tau) = \left[\frac{1 + n_{\text{eq}}(1-\xi)}{1 + n_{\text{eq}}(1-x)}\right]^{1+l(n_{\text{eq}}+1)} \left(\frac{1-x}{1-\xi}\right)^{l n_{\text{eq}}} g(\xi,\, 0)\,. \tag{A9}$$

The function $\xi(x, \tau)$ follows from inversion of Eq. (A7), and this is

$$\xi(x,\, \tau) = \frac{1 + (x-1)[(n_{\text{eq}}+1)\exp(-\tau) - n_{\text{eq}}]}{1 + n_{\text{eq}}(1-x)[1 - \exp(-\tau)]}\,. \tag{A10}$$

When we eliminate ξ in favor of τ in Eq. (A9) we obtain

$$g(x,\, \tau) = \frac{\exp(n_{\text{eq}} l \tau)}{\{1 + n_{\text{eq}}(1-x)[1 - \exp(-\tau)]\}^{l+1}}\, g(\xi,\, 0)\,, \tag{A11}$$

which is Eq. (2.5).

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