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Time evolution of radiation in a damped cavity

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Abstract. Any initial state of the radiation field in a finite-Q cavity relaxes towards thermal equilibrium. For the case of a zero-temperature cavity, it is shown that the factorial moments of any photon distribution evolve independently, and have the same time evolution. This is applied to evaluate the transient behaviour of the photon statistics for a variety of initial states of practical interest. The coherences of the density operator obey the same equation of motion as the populations (the photon probability distribution), after a simple transformation. This is used to recover the result that a coherent state remains a pure and coherent state at all times. Also the time-dependent frequency spectrum has been evaluated, and it appeared that this spectrum is the same for any state of the radiation field. The spectral distribution is determined entirely by the cavity damping rate, the spectral width of the detector, and the observation time.

1. Introduction

Radiation in a single-mode optical cavity is probably the simplest quantum-mechanical system, but is still of current interest, in particular with respect to the generation of squeezed states. The probability to find n photons in the cavity at time t is determined by the density operator ρ of the radiation field according to

$$p_n(t) = \rho_{nn}(t) = \langle n|\rho(t)|n\rangle, \qquad n = 0, 1, 2, \ldots, \tag{1}$$

where $|n\rangle$ is a number state. The free evolution of the density operator is governed by the Liouvillian L_r, defined as

$$L_r\rho = \omega_c[a^\dagger a, \rho], \tag{2}$$

with ω_c the cavity frequency, and a and $a^\dagger$ the annihilation and creation operators, respectively. In a finite-Q cavity, radiation is absorbed by the mirrors, and this gives rise to damping in the time evolution of $\rho(t)$. This relaxation can be accounted for by a Liouvillian L_c, given by [1]

$$L_c\rho = \tfrac{1}{2}K(a^\dagger a\rho + \rho a^\dagger a - 2a\rho a^\dagger), \tag{3}$$

in the limit of zero temperature, and with $K = \omega_c/Q$. Then the equation of motion for $\rho(t)$ becomes

$$\mathrm{i}\frac{\mathrm{d}\rho}{\mathrm{d}t} = (L_r - \mathrm{i}L_c)\rho. \tag{4}$$

The effect of damping on the photon statistics of a squeezed state has been studied recently by Marian and Marian [2]. Of related interest is the problem of a two-state atom in a single-mode cavity, e.g., the Jaynes–Cummings model. When atoms are injected into a cavity, then during the time in between the passages of two atoms, the

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radiation field evolves according to (4). In the presence of an atom, the equation of motion has to be integrated numerically [3]. The effects of cavity damping in the Jaynes–Cummings model has been studied extensively [4–14].

2. Probability distribution

By taking diagonal matrix elements with respect to number states in the equation of motion (4) we obtain

$$\frac{\mathrm{d}p_n}{\mathrm{d}t}=K[(n+1)p_{n+1}-np_n]. \tag{5}$$

In terms of the dimensionless time τ:

$$\tau=Kt, \tag{6}$$

this becomes

$$\frac{\mathrm{d}p_n}{\mathrm{d}\tau}=(n+1)p_{n+1}-np_n. \tag{7}$$

This equation can be solved by Laplace transform, and the solution is well known [2, 15]. For applications later on, we would like to give a simplified derivation of this solution. When we multiply (7) by $n!/(n-k)!$, with $k=0,1,2,\ldots$ but fixed, use the convention that the factorial of a negative integer is infinite, and then sum over n, we find

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\langle\frac{n!}{(n-k)!}\right\rangle=-k\left\langle\frac{n!}{(n-k)!}\right\rangle. \tag{8}$$

The factorial moments $s_k(\tau)$ of the probability distribution $p_n(\tau)$ are defined as

$$s_k(\tau)=\sum_{n=0}^{\infty}\frac{n!}{(n-k)!}p_n(\tau)=\left\langle\frac{n!}{(n-k)!}\right\rangle(\tau), \qquad k=0,1,2,\ldots, \tag{9}$$

and with (8) it then follows that the factorial moments obey the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}\tau}s_k(\tau)=-ks_k(\tau). \tag{10}$$

Equation (7) for p_n couples the probabilities with different n values, but in (10) the factorial moments for different k evolve independently. In this sense, the transformation to factorial moments diagonalizes (7).

The solution of (10) is

$$s_k(\tau)=\exp(-k\tau)s_k(0), \tag{11}$$

in terms of the initial factorial moments $s_k(0)$. This has been found before in a different way [2]. For a given initial state $\rho(0)$ of the radiation field, the probabilities $p_n(0)$ follow from (1) with $\tau=0$. Then the initial factorial moments are determined by (9) with $\tau=0$, after which (11) gives the factorial moments for all times. The probabilities as a function of time follow from the inverse of (9), which is in general

$$p_n(\tau)=\frac{1}{n!}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}s_{n+k}(\tau), \tag{12}$$

as can be checked by inspection. Finally, with solution (11) this can be expressed in terms of the factorial moments $s_k(0)$, and with (9) with $\tau=0$ we can express this again in terms of the initial probabilities. The result is

$$p_n(\tau)=\sum_{m=n}^{\infty}\binom{m}{n}\exp(-n\tau)\,[1-\exp(-\tau)]^{m-n}p_m(0). \tag{13}$$

This probability distribution $p_n(\tau)$ is a Bernoulli distribution over the initial distribution $p_n(0)$, and it has the interpretation that each photon that was in the cavity at time $\tau=0$ has a probability of $\exp(-\tau)$ for still being in the cavity at time τ, as is well-known [15].

For τ large, e.g., $t\gg 1/K$, this reduces to

$$p_n(\infty)=\delta_{n,0}, \tag{14}$$

which is the thermal-equilibrium distribution at zero temperature, corresponding to no photons left in the cavity. The corresponding factorial moments are

$$s_k(\infty)=\delta_{k,0}. \tag{15}$$

Notice that for $k=0$ we have

$$s_0(\tau)=\sum_{n=0}^{\infty}p_n(\tau)=\mathrm{Tr}\,\rho(\tau)=1, \tag{16}$$

which is nothing but the normalization of the density operator.

3. Average, variance, and q factor

The average number of photons in the cavity at time τ is

$$\bar{n}(\tau)=\langle n\rangle(\tau)=s_1(\tau), \tag{17}$$

and with (11) this becomes

$$\bar{n}(\tau)=\exp(-\tau)\,\bar{n}(0). \tag{18}$$

The variance of the distribution can be expressed in terms of the factorial moments as

$$\mathrm{var}(\tau)=\langle n^2\rangle(\tau)-\langle n\rangle(\tau)^2=s_2(\tau)+s_1(\tau)-s_1(\tau)^2, \tag{19}$$

which can be evaluated with (11). The normalized variance is Mandel's q factor

$$q(\tau)=\frac{\mathrm{var}(\tau)-\bar{n}(\tau)}{\bar{n}(\tau)}=\frac{s_2(\tau)-s_1(\tau)^2}{s_1(\tau)}, \tag{20}$$

and with (11) this becomes

$$q(\tau)=\exp(-\tau)\,q(0). \tag{21}$$

With $\tau=Kt$ this shows that the q factor decays with a time constant of $1/K$ for any initial distribution of the radiation. It also implies that a sub-Poisson distribution ($q<0$) remains sub-Poisson for all times.

4. Special cases

4.1. *Number state*

When the initial state is a number state with exactly $n_0 = 0, 1, 2, \ldots$ photons in the cavity, then the initial probability distribution is given by

$$p_n(0) = \delta_{n, n_0}, \tag{22}$$

and the initial factorial moments are

$$s_k(0) = \frac{n_0!}{(n_0 - k)!}. \tag{23}$$

This gives $\bar{n}(0) = n_0$ and $q(0) = -1$. The time-dependent factorial moments are then given by (11), and the τ-dependent probability distribution becomes

$$p_n(\tau) = \binom{n_0}{n} \exp(-n\tau)\,[1 - \exp(-\tau)]^{(n_0 - n)}. \tag{24}$$

This is a binomial or Bernoulli distribution, as could be expected. Notice that $p_n(\tau) = 0$ for $n > n_0$, as it should be, due to the binomial coefficient.

4.2. *Coherent state*

For a coherent state $|\alpha\rangle$, α complex, as initial state, the probability distribution is

$$p_n(0) = \frac{|\alpha|^{2n}}{n!} \exp(-|\alpha|^2), \tag{25}$$

which gives for the factorial moments

$$s_k(0) = |\alpha|^{2k}. \tag{26}$$

This is a Poisson distribution with $\bar{n}(0) = |\alpha|^2$ and $q(0) = 0$. The time-dependent probabilities then are

$$p_n(\tau) = \frac{|\alpha|^{2n}}{n!} \exp[-n\tau - |\alpha|^2 \exp(-\tau)], \tag{27}$$

and this is again a Poisson distribution, with effectively

$$|\alpha|^2 \to |\alpha|^2 \exp(-\tau). \tag{28}$$

4.3. *Binomial distribution*

The binomial distribution, as initial state, is defined as

$$p_n(0) = \binom{n_0}{n} \beta^n (1 - \beta)^{n_0 - n}, \tag{29}$$

with β and $n_0 = 0, 1, 2, \ldots$ as free parameters. The values of β are restricted by $0 \leqslant \beta \leqslant 1$. The average number of photons is $\bar{n}(0) = \beta n_0$ and the q factor is $q(0) = -\beta$. Consequently, this distribution has sub-Poisson statistics. It has furthermore the interesting property that it interpolates between a number state and a coherent state. For $\beta = 1$ this is the distribution for the number state with n_0 photons. For $\beta \to 0$, $n_0 \to \infty$, with the product βn_0 remaining finite, this is the Poisson distribution of the coherent state, with $|\alpha|^2=\beta n_0$. With (9) we obtain for the factorial moments of the initial distribution

$$s_k(0)=\beta^k\frac{n_0!}{(n_0-k)!}, \tag{30}$$

and with (11) we then obtain

$$s_k(\tau)=\beta^k\exp(-k\tau)\frac{n_0!}{(n_0-k)!}. \tag{31}$$

This is again a binomial distribution, with effectively

$$\beta\to\beta\exp(-\tau), \tag{32}$$

and hence the time-dependent probability distribution is given by

$$p_n(\tau)=\binom{n_0}{n}\beta^n\exp(-n\tau)[1-\beta\exp(-\tau)]^{n_0-n}. \tag{33}$$

This illustrates the computational advantage of the use of factorial moments, rather than relation (13) directly.

4.4. *Thermal field*

For a finite temperature thermal state as initial distribution we have [16]

$$p_n(0)=\frac{n_t^n}{(n_t+1)^{n+1}}, \tag{34}$$

with $n_t>0$ as free parameter. For this distribution we have $\bar{n}(0)=q(0)=n_t$, and the factorial moments are

$$s_k(0)=k!n_t^k. \tag{35}$$

With (11) we then find

$$s_k(\tau)=k![n_t\exp(-\tau)]^k, \tag{36}$$

and consequently this is again a thermal distribution, with effectively

$$n_t\to n_t\exp(-\tau). \tag{37}$$

The probability distribution therefore becomes

$$p_n(\tau)=\frac{n_t^n\exp(-n\tau)}{[n_t\exp(-\tau)+1]^{n+1}}. \tag{38}$$

4.5. *Negative binomial distribution*

An interpolation between a coherent state and a thermal state is provided by the negative binomial distribution [17, 18]. Such states can be generated in parametric amplification [19]. In terms of the two free parameters d and ζ, with $d\geqslant 1$ and $0<\zeta<1$, the initial distribution is given by

$$p_n(0)=\binom{n+d-1}{n}\zeta^d(1-\zeta)^n. \tag{39}$$

The average number of photons is

$$\bar{n}(0)=d\left(\frac{1}{\zeta}-1\right). \tag{40}$$

Alternatively, we can consider $n_a=\bar{n}(0)>0$ instead of ζ as independent parameter, in terms of which the probability distribution becomes

$$p_n(0)=d^d\frac{\Gamma(n+d)}{n!\Gamma(d)}\frac{n_a^n}{(n_a+d)^{n+d}}. \tag{41}$$

For $d=1$ this is the thermal distribution, and in the limit $d\to\infty$ this becomes the distribution of a coherent state. The factorial moments are

$$s_k(0)=d(d+1)\ldots(d+k-1)\left(\frac{n_a}{d}\right)^k, \qquad k=1,2,\ldots, \tag{42}$$

and $s_0=1$. From (42) and (20) we find $q(0)=n_a/d>0$, and therefore the statistics are super-Poissonian. Then the time-dependent factorial moments become

$$s_k(\tau)=d(d+1)\ldots(d+k-1)\left(\frac{n_a}{d}\right)^k\exp(-k\tau), \tag{43}$$

which shows that the τ-dependence effectively gives

$$n_a\to n_a\exp(-\tau). \tag{44}$$

Therefore, the probability distribution is

$$p_n(\tau)=d^d\frac{\Gamma(n+d)}{n!\Gamma(d)}\frac{n_a^n\exp(-n\tau)}{[n_a\exp(-\tau)+d]^{n+d}}. \tag{45}$$

4.6. *Non-central negative binomial distribution*

A superposition of a coherent state and a thermal state is a non-central negative binomial distribution [20]. Such states correspond to the output of a linear amplifier when the input is a coherent state [21, 22]. This distribution has as free parameters $n_c>0$, $n_t>0$, and $d\geqslant 1$, and is given by

$$p_n(0)=\frac{d^d n_t^n}{(d+n_t)^{n+d}}\exp\left(-\frac{n_c}{n_t}d\right)\frac{1}{n!}\sum_{m=0}^{\infty}\frac{\Gamma(m+n+d)}{m!\Gamma(m+d)}\left[\frac{n_c d^2}{n_t(d+n_t)}\right]^m. \tag{46}$$

For $n_c=0$ only the $m=0$ term survives, and this reduces to the negative binomial distribution with $n_a=n_t$. For $n_c=0$ and $d=1$ this is the thermal distribution with n_t photons, and for $n_t=0$ it reduces to the coherent distribution with $|\alpha|^2=n_c$, irrespective of the value of d. The average number of photons is $\bar{n}(0)=n_c+n_t$ and the q factor turns out to be

$$q(0)=\frac{n_t}{d}\frac{n_t+2n_c}{n_t+n_c}>0, \tag{47}$$

corresponding to super-Poisson statistics. With equation (9) we can evaluate the initial factorial moments, which gives after some manipulations

$$s_k(0)=\exp\left(-\frac{n_c}{n_t}d\right)\left(\frac{n_t}{d}\right)^k\sum_{m=0}^{\infty}\frac{\Gamma(m+d+k)}{m!\Gamma(m+d)}\left(\frac{n_c d}{n_t}\right)^m. \tag{48}$$

The series in (46) and (47) can be expressed in terms of generalized Laguerre polynomials, defined by

$$L_n^{(\alpha)}(x)=\sum_{m=0}^{n}\frac{\Gamma(n+\alpha+1)}{(n-m)!\Gamma(m+\alpha+1)}\frac{(-x)^m}{m!},\qquad \alpha>-1,\qquad n=0,1,2,\ldots. \tag{49}$$

This yields

$$p_n(0)=\frac{d^d n_t^n}{(d+n_t)^{n+d}}\exp\left(-\frac{n_c d}{d+n_t}\right)L_n^{(d-1)}\left(-\frac{n_c d^2}{n_t(d+n_t)}\right), \tag{50}$$

$$s_k(0)=k!\left(\frac{n_t}{d}\right)^k L_k^{(d-1)}\left(-\frac{n_c}{n_t}d\right). \tag{51}$$

The time-dependent factorial moments then follow from (11), which gives

$$s_k(\tau)=k!\left[\frac{n_t\exp(-\tau)}{d}\right]^k L_k^{(d-1)}\left(-\frac{n_c}{n_t}d\right). \tag{52}$$

The time dependence is seen to give effectively the substitution

$$(n_c,n_t,d)\rightarrow[n_c\exp(-\tau),n_t\exp(-\tau),d], \tag{53}$$

in the initial state, and therefore the probability distribution is

$$p_n(\tau)=\frac{d^d n_t^n\exp(-n\tau)}{[d+n_t\exp(-\tau)]^{n+d}}\exp\left(-\frac{n_c d}{de^\tau+n_t}\right)L_n^{(d-1)}\left(-\frac{n_c d^2}{n_t[d+n_t\exp(-\tau)]}\right). \tag{54}$$

5. Coherences

When we take the $\langle n|\ldots|m\rangle$ matrix element of the equation of motion (4), then set $m=n+l$, and let again $\tau=Kt$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\rho_{n,n+l}=\left[\mathrm{i}\frac{\omega_c}{K}l-(n+\tfrac{1}{2}l)\right]\rho_{n,n+l}+[(n+1)(n+1+l)]^{1/2}\rho_{n+1,n+1+l}. \tag{55}$$

For $l=0$ this reduces to the equation of motion (7) for the probabilities $p_n(\tau)$. For $l\neq 0$ only the coherences along a diagonal in the density matrix couple together. In order to solve (55) we make the transformation

$$r_{n,l}(\tau)=\left[\frac{(n+l)!}{n!}\right]^{1/2}\exp\left[l\left(\frac{1}{2}-\mathrm{i}\frac{\omega_c}{K}\right)\tau\right]\rho_{n,n+l}(\tau), \tag{56}$$

which gives

$$\frac{\mathrm{d}r_{n,l}}{\mathrm{d}\tau}=(n+1)r_{n+1,l}-nr_{n,l}. \tag{57}$$

This equation of motion for $r_{n,l}(\tau)$ is identical in form to the equation of motion (7) for $p_n(\tau)$, for a given l, and therefore it has the same solution:

$$r_{n,l}(\tau)=\sum_{m=n}^{\infty}\binom{m}{n}\exp(-n\tau)[1-\exp(-\tau)]^{m-n}r_{m,l}(0). \tag{58}$$

Transforming back to the coherences then gives the solution

$$\rho_{n,n+l}(\tau)=\exp\left[\mathrm{i}l\frac{\omega_c}{K}\tau-(n+\tfrac{1}{2}l)\tau\right]\sum_{m=n}^{\infty}\frac{1}{(m-n)!}$$

$$\times[1-\exp(-\tau)]^{m-n}\left[\frac{m!(m+l)!}{n!(n+l)!}\right]^{1/2}\rho_{m,m+l}(0) \tag{59}$$

in terms of the initial coherences. This solution, in a slightly different form, has been found before by Laplace transformation and contour integration [2]. For $l=0$ equation (59) reduces to (13).

6. Pure states and mixtures

A state of the radiation field is a pure state if and only if $\mathrm{Tr}\,\rho^2=1$. When an initial state is prepared as a pure state, then due to the cavity damping it will evolve into a mixture, in general. For $\tau\to\infty$, any state becomes a pure state, since $\rho(\infty)=|0\rangle\langle 0|$, the vacuum, which is a pure state. With

$$\mathrm{Tr}\,\rho(\tau)^2=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}|\rho_{n,k}(\tau)|^2, \tag{60}$$

and the solution (59) for the coherences, this expression can be evaluated for a given initial state. It will not always be possible, however, to sum the double-series analytically.

As an example, consider the thermal distribution for which the density operator is diagonal:

$$\rho(\tau)=\sum_{n=0}^{\infty}|n\rangle p_n(\tau)\langle n|, \tag{61}$$

with $p_n(\tau)$ given by (38). We then find

$$\mathrm{Tr}\,\rho(\tau)^2=\sum_{n=0}^{\infty}p_n(\tau)^2=\frac{1}{1+2n_t\exp(-\tau)}. \tag{62}$$

The right-hand side increases monotonically from $1/(1+2n_t)$ to unity.

Another interesting example is the coherent state as initial state. Then the initial density-operator matrix elements are

$$\rho_{n,k}(0)=\frac{\alpha^n(\alpha^*)^k}{(n!k!)^{1/2}}\exp(-|\alpha|^2). \tag{63}$$

Then we let $k=n+l$, and substitute this into equation (59). Carrying out the summation then yields

$$\rho_{n,n+l}(\tau)=\frac{(\alpha^*)^l|\alpha|^{2n}}{[n!(n+l)!]^{1/2}}\exp\left[\mathrm{i}l\frac{\omega_c}{K}\tau-\left(n+\frac{1}{2}l\right)\tau-|\alpha|^2\exp(-\tau)\right]. \tag{64}$$

Then we set $l=k-n$, substitute this into (60), and evaluate the double sum. We then find

$$\mathrm{Tr}\,\rho(\tau)^2=1, \tag{65}$$

for all τ. This shows that a coherent state remains a pure state during the time evolution in a damped cavity. The state vector is

$$|\Psi(\tau)\rangle = \exp\left[-\tfrac{1}{2}|\alpha|^2 \exp(-\tau)\right] \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \exp\left(-in\frac{\omega_c}{K}\tau - \frac{1}{2}in\tau\right)|n\rangle, \tag{66}$$

because the corresponding density operator $\rho(\tau) = |\Psi(\tau)\rangle\langle\Psi(\tau)|$ has matrix elements given by (64). Furthermore, this state vector is a coherent state for all τ, with effectively

$$\alpha \to \alpha \exp\left(-\frac{1}{2}\tau - i\frac{\omega_c}{K}\tau\right). \tag{67}$$

7. Physical spectrum

The state of the radiation field in the cavity is essentially time dependent due to the damping, and therefore the frequency distribution of this radiation will depend on time. Let $E^{(+)}(t)$ be the positive-frequency part of the electric field operator in the Heisenberg picture at the location of a detector. In front of the detector is a frequency filter with setting ω, which transmits a filtered field given by

$$\bar{E}^{(+)}(t) = \int_0^{\infty} dt' \exp(-i\omega t') f(t') E^{(+)}(t-t'). \tag{68}$$

The function $f(t)$ depends on the filter, and is usually an exponential. In (68), the contribution of the free field, which is the modified vacuum near the spectrometer, has been neglected [23–25]. The time-dependent physical spectrum of light, as introduced by Eberly and Wódkiewicz [26], is then the photon counting rate by a detector in this filtered field:

$$I(\omega, t) = \xi\langle \bar{E}^{(-)}(t)\bar{E}^{(+)}(t)\rangle. \tag{69}$$

Here, ξ is an overall factor depending on the detector efficiency, the aperture, etc., and $\bar{E}^{(-)}$ is the Hermitian conjugate of $\bar{E}^{(+)}$. The physical spectrum can be written as [27]

$$I(\omega, t) = \int_0^{\infty} dt' \int_{-\infty}^{\infty} d\omega' s(\omega', t') J(\omega-\omega', t-t'), \tag{70}$$

with the smoothing function $s(\omega, t)$ defined by

$$s(\omega, t) = 2\,\mathrm{Re} \int_0^{\infty} dt' \exp(i\omega t') f(t) f(t+t')^*, \tag{71}$$

and the quasi- or Page–Lampard spectrum $J(\omega, t)$ given by [28, 29]

$$J(\omega, t) = \frac{\xi}{\pi}\,\mathrm{Re} \int_0^{\infty} dt' \exp(i\omega t') \langle E^{(-)}(t-t')E^{(+)}(t)\rangle. \tag{72}$$

In this way, $J(\omega, t)$ depends only on the field and $s(\omega, t)$ depends only on the detector.

A common choice for $f(t)$ is

$$f(t) = \gamma \exp(-\gamma t), \qquad \gamma > 0, \tag{73}$$

which gives for the smoothing function

$$s(\omega, t) = \frac{2\gamma^3 \exp(-2\gamma t)}{\gamma^2 + \omega^2}. \tag{74}$$

The frequency dependence is a Lorentzian around $\omega=0$ with half-width at half maximum (HWHM) equal to γ, and the function decays in time on a time scale of $1/\gamma$.

8. Field correlation function

We assume that the field in the cavity is prepared in a certain state at $t=0$, and that the detection of the field starts at this time. Then we have $\bar{E}^{(+)}(t)=0$ for $t<0$. Furthermore, $\bar{E}^{(+)}(t)\propto a(t)$, the annihilation operator, and this gives for the quasi spectrum

$$J(\omega, t)=\frac{1}{\pi}\operatorname{Re}\int_0^t \mathrm{d}t' \exp(\mathrm{i}\omega t')\langle a^\dagger(t-t')a(t)\rangle, \tag{75}$$

where some overall constants have been suppressed. The field correlation function $\langle a^\dagger(t-t')a(t)\rangle$ can be expressed in terms of Schrödinger operators as

$$\langle a^\dagger(t-t')a(t)\rangle=\operatorname{Tr} a\exp[-\mathrm{i}(L_r-\mathrm{i}L_c)t'][\rho(t-t')a^\dagger], \tag{76}$$

for $t'\geqslant 0$. Taking the trace gives

$$\langle a^\dagger(t-t')a(t)\rangle=\sum_{n=0}^{\infty}(n+1)^{1/2}\langle n+1|\{\exp[-\mathrm{i}(L_r-\mathrm{i}L_c)t'][\rho(t-t')a^\dagger]\}|n\rangle. \tag{77}$$

The equation of motion (4) for the density operator has the formal solution

$$\rho(t)=\exp[-\mathrm{i}(L_r-\mathrm{i}L_c)t]\,\rho(0), \tag{78}$$

which is governed by the same exponential of Liouvillians as the correlation function in (77). Therefore, the matrix element $\langle n+1|\ldots|n\rangle$ in (77) is related to the matrix elements of $[\rho(t-t')a^\dagger]$ in the same way as $\rho_{n+1,n}(t)$ is related to the matrix elements of $\rho(0)$. From (59) we find

$$\rho_{n+1,n}(t)=\exp[-\mathrm{i}\omega_c t-(n+\tfrac{1}{2})Kt]\sum_{m=n}^{\infty}\binom{m}{n}\times[1-\exp(-Kt)]^{m-n}\left(\frac{m+1}{n+1}\right)^{1/2}\rho_{m+1,m}(0), \tag{79}$$

which yields for the correlation function

$$\langle a^\dagger(t-t')a(t)\rangle=\exp(-\mathrm{i}\omega_c t'-\tfrac{1}{2}Kt')\times\sum_{n=0}^{\infty}\sum_{m=n}^{\infty}(m+1)\binom{m}{n}\exp(-nKt')[1-\exp(-Kt')]^{m-n}\rho_{m+1,m+1}(t-t'). \tag{80}$$

Changing the order of summation and summing over n then gives

$$\langle a^\dagger(t-t')a(t)\rangle=\exp(-\mathrm{i}\omega_c t'-\tfrac{1}{2}Kt')\sum_{m=0}^{\infty}mp_m(t-t'), \tag{81}$$

which simplifies to

$$\langle a^\dagger(t-t')a(t)\rangle=\exp(-\mathrm{i}\omega_c t'-\tfrac{1}{2}Kt')\,\bar{n}(t-t'). \tag{82}$$

With (18) for the average number of photons, this finally becomes

$$\langle a^\dagger(t-t')a(t)\rangle=\exp(-\mathrm{i}\omega_c t'-\tfrac{1}{2}Kt')\exp(-K(t-t'))\bar{n}(0). \tag{83}$$

This correlation function depends on the initial state only through the average number of photons at $t=0$, and is therefore essentially the same for any initial state. This implies that the spectral distribution is also the same for any initial state of the radiation field.

9. Quasi spectrum

Substituting (83) into (75) and carrying out the integration gives for the quasi spectrum

$$J(\omega, t)=\bar{n}(0)\exp(-\tfrac{1}{2}Kt)\frac{1}{\pi}\,\mathrm{Re}\frac{\exp[\mathrm{i}(\omega-\omega_c)t]-\exp(-\tfrac{1}{2}Kt)}{\mathrm{i}(\omega-\omega_c)+\tfrac{1}{2}K}. \tag{84}$$

The strength of this spectrum is

$$\int_{-\infty}^{\infty}\mathrm{d}\omega\, J(\omega, t)=\begin{cases}0 & t=0,\\ \bar{n}(0)\exp(-Kt), & t>0.\end{cases} \tag{85}$$

For $t=0$ the strength is zero because $J(\omega, 0)=0$. For $t>0$ the strength is equal to the average number of photons in the cavity at time t. Again we set $\tau=Kt$, and then we introduce the dimensionless frequency, relative to the cavity frequency:

$$\lambda=\frac{\omega-\omega_c}{K}, \tag{86}$$

and the normalized dimensionless profile:

$$\hat{J}(\lambda, \tau)=\frac{\pi K}{\bar{n}(0)}J(\omega, t). \tag{87}$$

This gives:

$$\hat{J}(\lambda, \tau)=\frac{2\exp(-\tfrac{1}{2}\tau)}{1+4\lambda^2}[2\lambda\sin(\lambda\tau)+\cos(\lambda\tau)-\exp(-\tfrac{1}{2}\tau)]. \tag{88}$$

The spectrum is symmetric around $\lambda=0$:

$$\hat{J}(-\lambda, \tau)=\hat{J}(\lambda, \tau), \tag{89}$$

and its peak value at $\lambda=0$ is

$$\hat{J}(0, \tau)=2[\exp(-\tfrac{1}{2}\tau)-\exp(-\tau)]. \tag{90}$$

This peak is maximum for $\tau=2\ln 2$, with a value of $\hat{J}(0, 2\ln 2)=\tfrac{1}{2}$. Figure 1 illustrates the behaviour of the quasi spectrum for three values of τ. Notice that this spectrum becomes negative for certain frequencies. For τ small the spectrum is very broad, but it narrows with increasing τ.

10. Spectral distribution

With (84) for $J(\omega, t)$ and (74) for $s(\omega, t)$ the physical spectrum $I(\omega, t)$ in (70) can be obtained. The ω' integral can be evaluated by contour integration, after which the t' integral only involves simple exponentials. The result is

$$I(\omega, t)=2\gamma^2\bar{n}(0)\,\mathrm{Re}\frac{\exp(-2\gamma t)}{\tfrac{1}{2}K-\gamma+\mathrm{i}(\omega-\omega_c)}$$
$$\times\left\{\frac{\exp\{[\gamma-\tfrac{1}{2}K+\mathrm{i}(\omega-\omega_c)]t\}-1}{\gamma-\tfrac{1}{2}K+\mathrm{i}(\omega-\omega_c)}-\frac{\exp\{2\gamma-K)t\}-1}{2\gamma-K}\right\}. \tag{91}$$

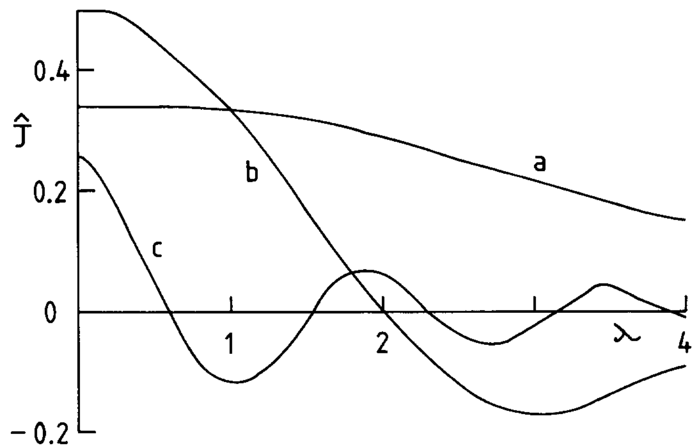

Figure 1. Plot of the normalized quasi spectrum $\hat{J}(\lambda, \tau)$ as a function of λ, for $\lambda \geqslant 0$ (the spectrum is symmetric). Curves a, b and c correspond to $\tau = 0{\cdot}5$, $2\ln 2$, and 4, respectively. The oscillatory behaviour increases with τ, and the spectrum narrows with τ.

For the strength of the physical spectrum we obtain

$$\int_{-\infty}^{\infty} d\omega\, I(\omega, t) = 2\pi\bar{n}(0)\frac{\gamma^2}{K - 2\gamma}[\exp(-2\gamma t) - \exp(-Kt)], \qquad t \geqslant 0. \tag{92}$$

Unlike for the quasi spectrum, the strength of the physical spectrum is continuous at $t = 0$. Furthermore, there is no discontinuity at $\gamma = \frac{1}{2}K$.

Then we express the spectrum in terms of the dimensionless parameters τ and λ, introduce the dimensionless frequency width $\hat{\gamma}$ of the spectrometer:

$$\hat{\gamma} = \gamma/K, \tag{93}$$

and normalize $I(\omega, t)$ as

$$\hat{I}(\lambda, \tau) = \frac{K^2}{2\gamma^2\bar{n}(0)} I(\omega, t). \tag{94}$$

Working out the real part then gives

$$\hat{I}(\lambda, \tau) = \frac{\frac{1}{2}}{(\hat{\gamma} - \frac{1}{2})^2 + \lambda^2}[\exp(-2\hat{\gamma}\tau) + \exp(-\tau) - 2\cos(\lambda\tau)\exp(-(\hat{\gamma} + \tfrac{1}{2})\tau)]. \tag{95}$$

Figure 2 shows the behaviour of $\hat{I}(\lambda, \tau)$ for three values of τ. The physical spectrum is, of course, positive for all λ. For small τ also the physical spectrum is very broad, and it narrows with increasing τ. The broadening is due to the cavity width K, the spectrometer width γ, and the finite detection time. For $\hat{\gamma} \to 0$ and $\tau \to \infty$ the spectrum reduces to

$$\lim_{\hat{\gamma} \to 0} \hat{I}(\lambda, \tau = \infty) = \frac{2}{1 + 4\lambda^2}, \tag{96}$$

a Lorentzian with HWHM $= 1/2$. Figure 3 illustrates the spectrum when the frequency resolution of the detector is perfect ($\hat{\gamma} \to 0$), and it is seen that the width is still much larger than a Lorentzian of the same height, due to the finite detection time.

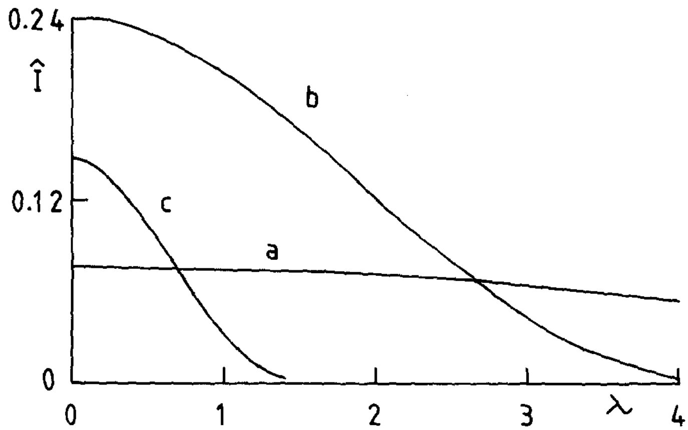

Figure 2. Plot of the normalized physical spectrum for $\hat{\gamma}=\frac{1}{2}$, and the three curves correspond to the same τ values as in figure 1. The physical spectrum is also symmetric around $\lambda=0$, and narrows with increasing τ.

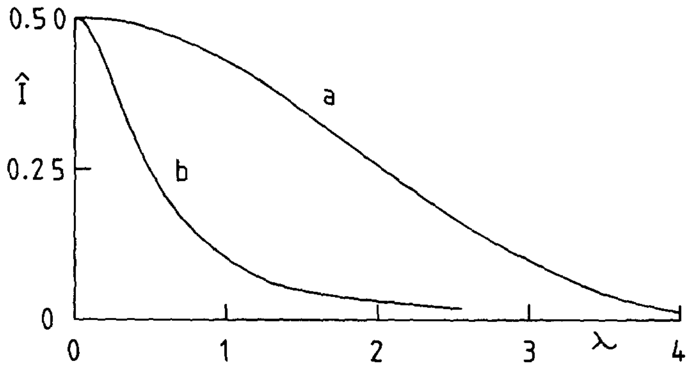

Figure 3. Curve *a* shows the physical spectrum for perfect frequency resolution ($\hat{\gamma}=0$) and $\tau=2\ln 2$. Curve *b* is a Lorentzian with the same height and with a width equal to $\frac{1}{2}$, corresponding to the physical spectrum for $\tau\to\infty$. It appears that the finite detection time gives rise to a significant broadening of the spectral distribution.

11. Conclusions

We have studied some aspects of the temporal evolution of radiation in a zero-temperature damped cavity. It was shown that the use of factorial moments greatly facilitates the evaluation of the photon probability distribution, as was illustrated with some practical examples. The photon probability distribution of the radiation inside the cavity determines the photon count distribution which can be measured by a detector, although in a non-trivial way. This will be shown elsewhere. With the result for the coherences of the density operator, it was shown that a coherent state remains a coherent state at all times, although with a diminishing value of α due to the cavity damping. This result has been obtained before (p. 399 of [1]) by solving the Fokker–Planck equation corresponding to the Liouville equation (4). The frequency distribution of the radiation, expressed in terms of the physical spectrum, turned out to be independent of the initial state of the radiation. The spectral distribution was evaluated explicitly, and it was shown that the finite observation time strongly broadens the spectrum.

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