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Light scattering by a phase conjugator in the four-wave mixing configuration

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Abstract. Reflection of travelling and evanescent plane waves by a four-wave mixing phase-conjugator is studied in detail. No restrictions are imposed on the nonlinear interaction strength, the angle of incidence or the frequency mismatch between the pump-beams and the incoming waves. We only assume that the incident field is weak compared to the pump fields, which justifies a classical field description of the pumps. The wave-vectors, amplitudes and phases for the various waves are evaluated, without the slowly-varying amplitude approximation. Familiar phase-matching resonances for certain values of the interaction length are recovered, and in addition strong resonances are found if the angle of incidence is finite and the incident light is not in perfect resonance with the pumps. The latter resonances appear at the transitions from a travelling to an evanescent wave. The significance of finite angles of incidence and evanescent waves for spectroscopic applications is pointed out.

1. Introduction

Among the many ways [1–3] of generating a phase-conjugated signal with respect to a reference signal, the technique of four-wave mixing is the most promising from an experimental point of view [4–9]. A nonlinear crystal (like $BaTiO_3$) or liquid (typically CS_2) is irradiated by two counter-propagating strong laser beams (the pumps) with intensity I. A third incident (weak) field then couples to the pump fields through the third-order susceptibility $\chi^{(3)}$, and the result is an electric polarization of the medium, which is proportional to $\chi^{(3)}I$ and the electric field component of the weak field. This induced polarization then emits radiation which propagates out of the crystal. Under certain conditions this generated fourth wave is the phase-conjugated, or time-reversed, replica of the incident field. Production of phase-conjugated radiation is of great practical importance in optical engineering, because it provides a method for correction of wavefront distortions.

In most applications the weak field is a nearly monochromatic plane wave with well defined polarization, the angle of incidence on the crystal is almost zero (usually a few degrees), and the coupling constant $\gamma \propto \chi^{(3)}I$ is very small. For this configuration the generation of phase-conjugated waves is well understood [10–19]. There are, however, conceivable applications in which these conditions do not hold.

It has been predicted, for instance, that the lifetime of an atom in the neighbourhood of a phase conjugator (PC) is infinite, as a result of the fact that an ideal PC focuses the emitted fluorescence exactly back on the atom [20, 21]. Consequently, the spectroscopic linewidth of the atomic transition under consideration would be zero, which can have a great impact on frequency standards. These predictions were derived under the assumption of perfect phase-conjugation. Since phase-conjugation is equivalent to time-reversal, it is obvious that ideal PCs cannot exist (owing to violation of causality). Nevertheless, it can be anticipated that realistic PCs may possibly be used to manipulate linewidths over a large range (in contrast to the situation of atoms near a metal surface, where the change in lifetime is at best a factor of two).

Emitted dipole radiation (fluorescence) by an atom in the vicinity of a PC has plane-wave components, which are incident on the surface at every angle of incidence. Besides that, a dipole field has evanescent components (exponentially-decaying waves)†, and the radiation is not monochromatic. Furthermore, for contemporary high-power lasers the interaction parameter $\gamma \propto I$ is not necessarily small. In this paper, we present a general treatment of the scattering of travelling and evanescent waves by a four-wave mixing PC, without restrictions on the interaction strength, frequency, polarization or angle of incidence.

2. The model

A nonlinear transparent crystal ($\chi^{(3)} \neq 0$, $\chi^{(i \neq 3)} = 0$) occupies the region $0 > z > -\Delta$, $\Delta > 0$, in an xyz Cartesian coordinate frame, and the regions $z > 0$ and $z < -\Delta$ are empty space. Two counter-propagating pump-beams with intensity I and frequency $\bar{\omega}$ (called the setting frequency of the PC) illuminate the medium. Then the complex-valued coupling parameter is given by $\gamma \propto \chi^{(3)} I$. We shall assume that I is constant (no depletion of pumps) and that $\chi^{(3)}$ is frequency independent. This means that the incident probe field must have a bandwidth around $\bar{\omega}$ which is smaller than the frequency width of $\chi^{(3)}$. In this fashion we can avoid complicated notations, but it is straightforward to retain the frequency dependence of γ if necessary [22]. Furthermore, we assume that the tensorial nature of $\chi^{(3)}$ is irrelevant, which can always be managed by a proper choice of geometry.

The radiation field shall be represented by its electric and magnetic components, $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ respectively, and charges and currents by a polarization density $\mathbf{P}(\mathbf{r}, t)$. It is advantageous to adopt a Fourier transform with respect to time,

$$\hat{\mathbf{E}}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} \exp(\mathrm{i}\omega t) \mathbf{E}(\mathbf{r}, t)\, \mathrm{d}t, \quad \omega \text{ real}, \tag{1}$$

and since $\mathbf{E}(\mathbf{r}, t)$ is real we have

$$\hat{\mathbf{E}}(\mathbf{r}, -\omega) = \hat{\mathbf{E}}(\mathbf{r}, \omega)^*. \tag{2}$$

† According to Fourier's theorem, every field can be expanded in terms of travelling plane waves. Maxwell's equations, however, impose the restriction that the field must be transverse. Even when a radiation field is transverse, its Fourier expansion can acquire longitudinal components which do not obey Maxwell's equations individually. This complication can be avoided by allowing the field to consist of travelling and evanescent transverse plane waves.

Similar notations apply to $\mathbf{B}(\mathbf{r}, t)$ and $\mathbf{P}(\mathbf{r}, t)$. In the Fourier domain, Maxwell's equations read

$$\left.\begin{aligned} &\nabla\cdot(\hat{\mathbf{E}}+\hat{\mathbf{P}}/\varepsilon_0)=0, \qquad \nabla\cdot\hat{\mathbf{B}}=0, \\ &\nabla\times\hat{\mathbf{E}}=\mathrm{i}\omega\hat{\mathbf{B}}, \qquad \nabla\times\hat{\mathbf{B}}=\frac{-\mathrm{i}\omega}{c^2}(\hat{\mathbf{E}}+\hat{\mathbf{P}}/\varepsilon_0), \end{aligned}\right\} \tag{3}$$

which should hold for all $\mathbf{r}$ and ω.

In the regions $z>0$ and $z<-\Delta$ the polarization density $\hat{\mathbf{P}}$ is zero. Inside the nonlinear medium $\hat{\mathbf{P}}(\mathbf{r}, \omega)$ is proportional to the electric field at a different frequency. Explicitly [23], this has the form

$$\hat{\mathbf{P}}(\mathbf{r},\omega)=\begin{cases}\varepsilon_0\gamma^*\hat{\mathbf{E}}(\mathbf{r},\omega-2\bar{\omega}), & \omega>0,\\ \varepsilon_0\gamma\hat{\mathbf{E}}(\mathbf{r},2\bar{\omega}+\omega), & \omega<0,\end{cases} \tag{4}$$

where the field $\hat{\mathbf{E}}$ does not include the two pump fields. These are parametrically accounted for by $\gamma\propto I$. On the surfaces $z=0$ and $z=-\Delta$, Maxwell's equations (3) imply the usual boundary conditions.

3. Dispersion relation

Before we can solve the scattering problem for travelling and evanescent waves by this PC, we have to establish the fundamental plane-wave solutions which are supported by the medium. To this end we first notice that the polarization density $\hat{\mathbf{P}}(\mathbf{r}, \omega)$ from equation (4) couples positive and negative frequencies. If we denote by $\omega_1\approx\bar{\omega}>0$ a fixed positive frequency, then $\hat{\mathbf{P}}(\mathbf{r}, \omega_1)$ is determined by the electric field component with frequency

$$\omega_2=\omega_1-2\bar{\omega}. \tag{5}$$

The polarization at this negative frequency $\omega_2\approx-\bar{\omega}$ is then proportional to the electric field at $2\bar{\omega}+\omega_2=\omega_1$, according to equation (4). Hence the nonlinear interaction couples positive and negative frequencies ω_1 and ω_2 in pairs. Consequently, for a fixed ω_1, Maxwell's equations (3) constitute essentially a set of eight equations, which have to be solved simultaneously.

The third Maxwell equation can be written as

$$\hat{\mathbf{B}}(\mathbf{r},\omega)=\frac{-\mathrm{i}}{\omega}\nabla\times\hat{\mathbf{E}}(\mathbf{r},\omega), \tag{6}$$

and therefore $\hat{\mathbf{B}}$ is known as soon as we have found $\hat{\mathbf{E}}$, both for ω_1 and ω_2. Then, $\nabla\cdot\hat{\mathbf{B}}=0$ is automatically satisfied. Furthermore, we notice that $\hat{\mathbf{P}}$ is proportional to $\hat{\mathbf{E}}$, although at a different frequency, and so the first Maxwell equation is certainly obeyed if

$$\nabla\cdot\hat{\mathbf{E}}(\mathbf{r},\omega)=0, \tag{7}$$

for every ω. Next we substitute equations (4) and (6) into the fourth Maxwell equation, which yields the set of coupled-wave equations

$$[\nabla^2+(\omega_1/c)^2]\hat{\mathbf{E}}(\mathbf{r},\omega_1)=-\gamma^*(\omega_1/c)^2\hat{\mathbf{E}}(\mathbf{r},\omega_2), \tag{8}$$

$$[\nabla^2+(\omega_2/c)^2]\hat{\mathbf{E}}(\mathbf{r},\omega_2)=-\gamma(\omega_2/c)^2\hat{\mathbf{E}}(\mathbf{r},\omega_1), \tag{9}$$

for the electric field. Equations (7)–(9) are the basic relations for a PC. The first one states that the fields are transverse, and (8) and (9) show that a wave with a positive frequency ω_1 couples to a wave with a negative frequency ω_2, and *vice versa*. Therefore, a positive-frequency field generates a negative-frequency field, which is the essence of a phase conjugator.

As a plane-wave solution, we try

$$\hat{\mathbf{E}}_a(\mathbf{r}, \omega_1) = \mathbf{E}_a \exp(\mathrm{i}\mathbf{k}_a \cdot \mathbf{r}), \tag{10}$$

$$\hat{\mathbf{E}}_a(\mathbf{r}, \omega_2) = \eta_a \mathbf{E}_a \exp(\mathrm{i}\mathbf{k}_a \cdot \mathbf{r}). \tag{11}$$

Then equations (7)–(9) give

$$\mathbf{k}_a \cdot \mathbf{E}_a = 0, \tag{12}$$

$$k_a^2 = k^2(1 + \gamma^* \eta_a) = k^2 \rho^2 (1 + \gamma/\eta_a), \tag{13}$$

where we have introduced the wavenumber

$$k = \omega_1/c,$$

and the dimensionless detuning parameter

$$\rho = 2\bar{\omega}/\omega_1 - 1. \tag{14}$$

Equation (12) states that plane waves in a PC are transverse, and the last equality in equation (13) gives an equation for the amplitude ratio η_a between the ω_2 and the ω_1 component. Because equation (13) is quadratic in η_a, it admits two solutions. For reasons which will become clear in due course, we choose the solution

$$\eta_a = \{\rho^2 - 1 - \delta[(\rho^2 - 1)^2 + 4\gamma_0^2 \rho^2]^{1/2}\}/2\gamma^*, \tag{15}$$

with

$$\gamma = \gamma_0 \exp(\mathrm{i}\phi), \quad \gamma_0 > 0, \quad \phi \text{ real}, \tag{16}$$

and

$$\delta = \operatorname{sgn}(\bar{\omega} - \omega_1) = \operatorname{sgn}(\rho - 1). \tag{17}$$

The solution corresponding to the second root for η_a will be written as

$$\hat{\mathbf{E}}_b(\mathbf{r}, \omega_1) = \eta_b \mathbf{E}_b \exp(\mathrm{i}\mathbf{k}_b \cdot \mathbf{r}), \tag{18}$$

$$\hat{\mathbf{E}}_b(\mathbf{r}, \omega_2) = \mathbf{E}_b \exp(\mathrm{i}\mathbf{k}_b \cdot \mathbf{r}), \tag{19}$$

which obeys Maxwell's equations if

$$\mathbf{k}_b \cdot \mathbf{E}_b = 0, \tag{20}$$

$$k_b^2 = k^2 \rho^2 (1 + \gamma \eta_b) = k^2 (1 + \gamma^*/\eta_b). \tag{21}$$

Of the two possible solutions for η_b we have to take

$$\eta_b = \{1 - \rho^2 + \delta[(\rho^2 - 1)^2 + 4\gamma_0^2 \rho^2]^{1/2}\}/2\gamma\rho^2. \tag{22}$$

With this convention we have $\eta_a \to 0$, $\eta_b \to 0$ if $\gamma \to 0$, and the a and b solutions become (uncoupled) ω_1 and ω_2 waves respectively in this limit. For $\gamma \neq 0$ the parameters η_a and η_b determine the relative strengths of the coupled waves with the complementary frequency, which are excited by the four-wave mixing process. Furthermore, we notice that the coupled waves $\hat{\mathbf{E}}_a(\mathbf{r}, \omega_1)$ and $\hat{\mathbf{E}}_a(\mathbf{r}, \omega_2)$ have the same wave-vector $\mathbf{k}_a$, which implies a perfect phase matching between these two waves at different frequencies. The same holds for the b solution.

Now we can substitute the expressions for η_a and η_b into equations (13) and (21), which gives the wavenumbers k_a and k_b (up to a minus sign). Notations can be simplified considerably by the introduction of the quantity

$$\varepsilon = \tfrac{1}{2}\{\rho^2 + 1 - \delta[(\rho^2 - 1)^2 + 4\gamma_0^2\rho^2]^{1/2}\}. \tag{23}$$

Then we find in terms of ε

$$k_a^2 = k^2\varepsilon, \qquad k_b^2 = k^2(\rho^2 + 1 - \varepsilon), \tag{24}$$

$$\eta_a = (\varepsilon - 1)/\gamma^*, \qquad \eta_b = (1 - \varepsilon)/\gamma\rho^2. \tag{25}$$

If the medium were an ordinary dielectric, we would also have $k_a^2 = k^2\varepsilon$, but with ε as the dielectric constant. An important difference is that in equation (23) ε depends explicitly on the frequency ω_1 through the parameter ρ. This frequency dependence is a genuine geometrical effect as it follows from the mechanism of four-wave mixing (rather than from a frequency dependence of $\chi^{(3)}$, which we have suppressed). Therefore, equation (24) gives the fundamental relations for the two branches of the dispersion curve for a four-wave mixing PC. This universal dispersion relation is plotted in figure 1. We remark that ε, as defined in equation (23) is real (possibly negative). If the frequency dependence of $\chi^{(3)}$ was retained ε could then also have an imaginary part. Furthermore, we notice that ε is discontinuous across the resonance $\omega_1 = \bar{\omega}$, or $\rho = 1$, due to the appearance of δ.

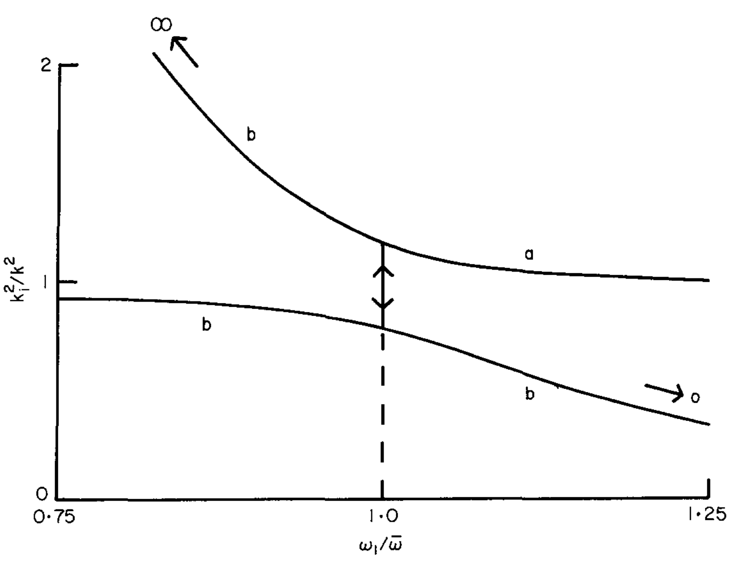

Figure 1. Dispersion relation for a phase conjugator. Curves a and b are k_a^2/k^2 and k_b^2/k^2 respectively, as a function of $\omega_1/\bar{\omega}$, and the coupling parameter is $\gamma_0 = 0{\cdot}2$. The discontinuous behaviour around $\omega_1 = \bar{\omega}$ comes from the choice of the roots in the definition of η_a and η_b.

4. Incident field

A given external field illuminates the surface $z=0$ of the PC. Since almost every field can be expanded in plane waves, and since Maxwell's equations for this PC are linear (even though the four-wave mixing process is not), it suffices to consider an incident field of the form

$$\hat{\mathbf{E}}_{\text{inc}}(\mathbf{r},\omega_1)=\mathbf{E}_{\text{inc}}\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}), \tag{26}$$

defined in the region $z>0$. The corresponding $\hat{\mathbf{B}}$ field follows from equation (6). In equation (26) the polarization and amplitude $\mathbf{E}_{\text{inc}}$, and the wave-vector $\mathbf{k}$ are arbitrary, with the restrictions that

$$\mathbf{k}\cdot\mathbf{E}_{\text{inc}}=0, \qquad k^2=\mathbf{k}\cdot\mathbf{k}=(\omega_1/c)^2, \tag{27}$$

according to Maxwell's equations in $z>0$.

With the unit vector $\mathbf{e}_z$ as the normal to the surface, we can decompose $\mathbf{k}$ into its parallel and perpendicular components with respect to the xy plane. We write

$$\mathbf{k}=\mathbf{k}_{\|}+k_z\mathbf{e}_z, \tag{28}$$

and similarly for any other vector quantity. Combining this with equation (27) gives

$$k_z^2=k^2-k_{\|}^2. \tag{29}$$

The quantity k^2 is a given positive number for a fixed ω_1, but the components $\mathbf{k}_{\|}$ and $k_z\mathbf{e}_z$ of $\mathbf{k}$ can be anything, as long as restriction (29) is satisfied. For most practical cases it is sufficient to consider only real-valued vectors $\mathbf{k}_{\|}$, and this will be assumed from now on. We shall regard the quantities $k=\omega_1/c>0$ and $\mathbf{k}_{\|}$ as given, but arbitrary. Then, the right-hand side of equation (29) is a given real number, and there are two possible solutions for k_z. Because the external field is generated by sources in the region $z>0$, we have to choose the causal solution, which is

$$k_z=\begin{cases}-(k^2-k_{\|}^2)^{1/2}, & \text{if } k>k_{\|},\\ -\mathrm{i}(k_{\|}^2-k^2)^{1/2}, & \text{if } k<k_{\|},\end{cases} \tag{30}$$

where $k_{\|}=(\mathbf{k}_{\|}\cdot\mathbf{k}_{\|})^{1/2}>0$. For $k>\mathrm{k}_{\|}$, k_z is negative and real, corresponding to an incident travelling plane wave from the region $z>0$. In the case $k<k_{\|}$, k_z is imaginary, and the root is chosen in such a way that the wave decays exponentially to zero in amplitude in the negative z direction. This evanescent wave decays in the direction perpendicular to the surface, and travels along the surface in the $\mathbf{k}_{\|}$ direction.

5. Fields

The incident field $\hat{\mathbf{E}}_{\text{inc}}(\mathbf{r},\omega_1)$ induces a nonlinear polarization in the medium, which in turn emits radiation according to the coupled-wave equations (8) and (9). This radiation travels out of the crystal and gives rise to reflected and transmitted waves by the layer. It will turn out that the fields everywhere in space can be expressed by plane travelling or evanescent waves, depending on k and $k_{\|}$. Every wave α will therefore have a spatial dependence of $\mathbf{E}_\alpha \exp(\mathrm{i}\mathbf{k}_\alpha \cdot \mathbf{r})$. Transversality then requires

$$\mathbf{k}_\alpha \cdot \mathbf{E}_\alpha = 0, \tag{31}$$

for every wave. Since the waves must be matched across the planes $z=0$ and $z=-\Delta$ with the aid of boundary conditions which should hold for all $\mathbf{r}$ in $z=0$ and $z=-\Delta$, all wave-vectors must have the same parallel component. Therefore, we have

$$\mathbf{k}_\alpha = \mathbf{k}_\parallel + k_{\alpha,z}\mathbf{e}_z, \tag{32}$$

and only $k_{\alpha,z}$ remains to be determined. On the other hand, the dispersion relations in vacuum and in the PC fix the value of $k_\alpha^2 = k_\parallel^2 + k_{\alpha,z}^2$. Consequently, the only freedom we have left is the choice of the sign of $k_{\alpha,z}$. In the regions $z>0$ and $z<-\Delta$ this sign is determined by the requirement that the waves must emanate from the PC, in the same way as we found the sign of k_z. For the fields inside the PC there is no *a priori* way to fix the signs of the z components of the wave-vectors, and therefore we have to retain all possible combinations. We shall only write down the expressions for the electric fields. Then, the magnetic fields can be found from equation (6).

5.1. *Region* z > 0

From the arguments above it follows that the most general plane-wave solution in the region $z>0$ is given by

$$\mathbf{E}(\mathbf{r}, \omega_1) = \mathbf{E}_{\mathrm{inc}} \exp(\mathrm{i}\mathbf{k} \cdot \mathbf{r}) + \mathbf{E}_{\mathrm{r}} \exp(\mathrm{i}\mathbf{k}_{\mathrm{r}} \cdot \mathbf{r}), \tag{33}$$

$$\hat{\mathbf{E}}(\mathbf{r}, \omega_2) = \mathbf{E}_{\mathrm{pc}} \exp(\mathrm{i}\mathbf{k}_{\mathrm{pc}} \cdot \mathbf{r}). \tag{34}$$

At frequency ω_1 there is only one other possible wave, which is the specularly-reflected r-wave with

$$k_{\mathrm{r}} = k, \qquad k_{\mathrm{r},z} = -k_z. \tag{35}$$

Although this field resembles that of a wave reflected by an ordinary dielectric, it is here entirely generated by the four-wave mixing process (we have set $\chi^{(1)}=0$). At ω_2 we have the phase-conjugated pc-wave with

$$k_{\mathrm{pc}} = -\omega_2/c > 0, \tag{36}$$

$$k_{\mathrm{pc},z}^2 = k_{\mathrm{pc}}^2 - k_\parallel^2 = k^2\rho^2 - k_\parallel^2. \tag{37}$$

Since ω_2 is negative, the pc-wave travels in the $-\mathbf{k}_{\mathrm{pc}}$ direction if $k_{\mathrm{pc},z}$ is real. In the case of an evanescent pc-wave, the wave should die out in the positive z direction. Consequently, the root should be taken as

$$k_{\mathrm{pc},z} = \begin{cases} -(k^2\rho^2 - k_\parallel^2)^{1/2}, & \text{if } k\rho > k_\parallel, \\ \mathrm{i}(k_\parallel^2 - k^2\rho^2)^{1/2}, & \text{if } k\rho < k_\parallel. \end{cases} \tag{38}$$

Then it remains to determine $\mathbf{E}_{\mathrm{r}}$ and $\mathbf{E}_{\mathrm{pc}}$.

5.2. *Region* 0 > z > −Δ

Inside the PC the fields are combinations of a and b solutions, and we have

$$\hat{\mathbf{E}}(\mathbf{r}, \omega_1) = \mathbf{E}_a^+ \exp(\mathrm{i}\mathbf{k}_a^+ \cdot \mathbf{r}) + \mathbf{E}_a^- \exp(\mathrm{i}\mathbf{k}_a^- \cdot \mathbf{r}) + \eta_b \mathbf{E}_b^+ \exp(\mathrm{i}\mathbf{k}_b^+ \cdot \mathbf{r}) + \eta_b \mathbf{E}_b^- \exp(\mathrm{i}\mathbf{k}_b^- \cdot \mathbf{r}), \tag{39}$$

$$\hat{\mathbf{E}}(\mathbf{r},\omega_2)=\eta_a\mathbf{E}_a^+\exp(\mathrm{i}\mathbf{k}_a^+\cdot\mathbf{r})+\eta_a\mathbf{E}_a^-\exp(\mathrm{i}\mathbf{k}_a^-\cdot\mathbf{r})+\mathbf{E}_b^+\exp(\mathrm{i}\mathbf{k}_b^+\cdot\mathbf{r})+\mathbf{E}_b^-\exp(\mathrm{i}\mathbf{k}_b^-\cdot\mathbf{r}). \tag{40}$$

The values of k_a^2 and k_b^2 are given in equation (24) and for the z components of the wave-vectors we write

$$k_{a,z}^{\pm}=\pm k_1, \tag{41}$$

$$k_{b,z}^{\pm}=\pm k_2, \tag{42}$$

with

$$k_1^2=k_a^2-k_\parallel^2, \tag{43}$$

$$k_2^2=k_b^2-k_\parallel^2. \tag{44}$$

The roots are taken (arbitrarily) as

$$k_1=\begin{cases}-(k_a^2-k_\parallel^2)^{1/2}, & \text{if } k_a^2>k_\parallel^2,\\ -\mathrm{i}(k_\parallel^2-k_a^2)^{1/2}, & \textit{if } k_a^2<k_\parallel^2,\end{cases} \tag{45}$$

$$k_2=\begin{cases}-(k_b^2-k_\parallel^2)^{1/2}, & \text{if } k_b^2>k_\parallel^2,\\ \mathrm{i}(k_\parallel^2-k_b^2)^{1/2}, & \text{if } k_b^2<k_\parallel^2.\end{cases} \tag{46}$$

5.3. *Region* $z<-\Delta$

The only waves which travel or die out in the negative z direction are

$$\hat{\mathbf{E}}(\mathbf{r},\omega_1)=\mathbf{E}_\mathrm{t}\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}), \tag{47}$$

$$\hat{\mathbf{E}}(\mathbf{r},\omega_2)=\mathbf{E}_\mathrm{nl}\exp(\mathrm{i}\mathbf{k}_\mathrm{nl}\cdot\mathbf{r}), \tag{48}$$

where the wave-vector of the transmitted (t) wave is the same as the incident wave-vector. Furthermore, there is possibly a nonlinear (nl) wave of frequency ω_2, which has

$$k_{\mathrm{nl},z}=-k_{\mathrm{pc},z}. \tag{49}$$

Figure 2 illustrates the various occurring waves.

6. Polarization and Fresnel coefficients

According to equation (32) every (complex-valued) wave-vector $\mathbf{k}_\alpha$ lies in the plane of incidence, spanned by the (real-valued) vectors $\mathbf{k}_\parallel$ and $\mathbf{e}_z$. The amplitude-polarization vectors $\mathbf{E}_\alpha$ can be decomposed into a surface (s) polarized and a plane (p) polarized component, which are perpendicular to and lie in the plane of incidence, respectively. Since $\mathbf{E}_\alpha$ is restricted by $\mathbf{k}_\alpha\cdot\mathbf{E}_\alpha=0$, the only ambiguity in a decomposition along unit s- and p polarization vectors is the choice of the phase of the unit vectors. We take

$$\mathbf{e}_{\alpha\mathrm{s}}=\frac{1}{k_\parallel}(\mathbf{k}_\parallel\times\mathbf{e}_z), \tag{50}$$

$$\mathbf{e}_{\alpha\mathrm{p}}=\frac{1}{k_\parallel k_\alpha}(k_{\alpha,z}\mathbf{k}_\parallel-k_\parallel^2\mathbf{e}_z), \tag{51}$$

and it is easy to check that $\mathbf{e}_{\alpha\mathrm{s}}$, $\mathbf{e}_{\alpha\mathrm{p}}$ and $\mathbf{k}_\alpha/k_\alpha$ constitute an orthonormal set of unit

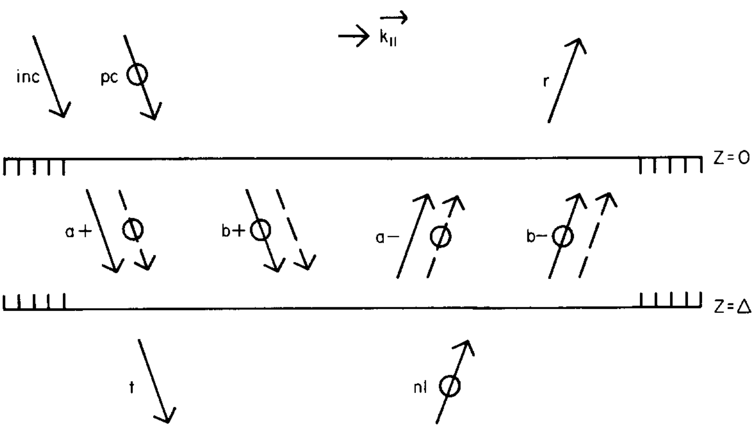

Figure 2. Schematic representation of the various waves from section 5. The arrows indicate the wave-vectors, which all have the same parallel component $\mathbf{k}_{\parallel}$. Their perpendicular components are approximately the same, apart from the sign. The inc-,t- and r waves are ω_1 waves and they travel in the direction of the wave-vector. The pc- and nl waves have a frequency $\omega_2 < 0$, and therefore they travel in the direction opposite to the wave-vector. This is indicated by a circle on the arrows. Inside the PC we have four different wave-vectors and every vector corresponds to two fields according to equations (39) and (40). The $a+$ and $a-$ fields are essentially ω_1-fields, and the $b+$ and $b-$ fields are negative-frequency waves. For $\gamma \neq 0$ these principle waves couple to a field with the same wave-vector but with a frequency of opposite sign. These fields are indicated by broken arrows. The wave which couples to the principle wave always propagates in the opposite direction.

vectors for every α. Notice that $\mathbf{e}_{\alpha s}$ is defined as independent of α and is real. The p-polarization vector can be complex. For all waves only the value of k_α^2 (real) is prescribed by the dispersion relation, and we take the square root as

$$k_\alpha = \begin{cases} (k_\alpha^2)^{1/2}, & \text{if } k_\alpha^2 > 0, \\ \mathrm{i}(-k_\alpha^2)^{1/2}, & \text{if } k_\alpha^2 < 0. \end{cases} \tag{52}$$

It can be proven that every wave is an s(p) wave if the incident wave is an s(p) wave. Therefore, we can distinguish between the two cases and write

$$\mathbf{E}_\alpha = E_\alpha \mathbf{e}_{\alpha\sigma}, \quad \sigma = \text{s or p}, \tag{53}$$

for every α. The simplification lies in the following relations for the z components of $\mathbf{E}_\alpha$

$$\left.\begin{aligned} &E_{\alpha,z} = 0, \quad \text{s waves}, \\ &E_{\alpha,z} = -\frac{k_{\parallel}}{k_\alpha} E_\alpha, \quad \text{p waves}. \end{aligned}\right\} \tag{54}$$

Then the Fresnel coefficients X_α(Y_α) for s(p) waves are defined as the ratio of E_α and $E_{\rm inc}$, or equivalently the amplitudes $\mathbf{E}_\alpha$ are written as

$$\left.\begin{aligned} \mathbf{E}_\alpha &= X_\alpha E_{\rm inc}\mathbf{e}_{\alpha s}, \quad \text{for s waves,} \\ \mathbf{E}_\alpha &= Y_\alpha E_{\rm inc}\mathbf{e}_{\alpha p}, \quad \text{for p waves.} \end{aligned}\right\} \tag{55}$$

The amplitude $E_{\rm inc}$ of the incident field is a given quantity, and the unit polarization vectors $\mathbf{e}_{\alpha\sigma}$ are geometrically determined. Therefore, knowledge of X_α and Y_α, for every wave α, determines the scattering of any wave by the PC.

7. Solution

Maxwell's equations (3) state that at the boundaries $z=0$ and $z=-\Delta$ the tangential component of $\mathbf{E}$, the normal component of $\mathbf{E}+\mathbf{P}/\varepsilon_0$, and the $\mathbf{B}$ field must be continuous, both for ω_1 and ω_2. In matching the fields in the three regions across the boundaries, we can determine all the $\mathbf{E}_\alpha$ values, and the results can be expressed in terms of the Fresnel coefficients X_α and Y_α for s- and p waves respectively. It is convenient to express the Fresnel coefficients in dimensionless quantities, rather than in wavenumbers. Besides ρ and ε we introduce

$$u=-k_z/k, \tag{56}$$

which allows us to write for the parallel components

$$k_\parallel^2=k^2(1-u^2). \tag{57}$$

For a travelling incident wave u is real, restricted by $0\leqslant u\leqslant 1$, and u equals the cosine of the angle of incidence. If the incident wave is evanescent, then u is positive imaginary. Furthermore, we define dimensionless wavenumbers by

$$m_\alpha=k_\alpha/k, \quad \alpha=a,b,1,2, \tag{58}$$

$$m_{\rm p}=k_{{\rm pc},z}/k, \tag{59}$$

and the layer thickness d in units of a wavelength of the incident radiation

$$d=k\Delta/2\pi. \tag{60}$$

Owing to boundary conditions at $z=-\Delta$, phase factors appear which can be expressed as

$$\left.\begin{aligned} \psi_1&=-2\pi d m_1, \qquad & \psi_2&=2\pi d m_2, \\ \psi_3&=2\pi d u, & \psi_4&=2\pi d m_{\rm p}. \end{aligned}\right\} \tag{61}$$

For travelling waves these phases are real, and for evanescent waves they are positive imaginary.

After laborious computations, it follows that the Fresnel coefficients can be expressed in terms of eight dimensionless parameters x_i, y_i, $i=1,\ldots,4$ as follows

$$
\begin{aligned}
&X_a^+=x_1, \qquad X_a^-=x_2,\\
&X_b^+=x_3, \qquad X_b^-=x_4,\\
&X_r=x_1+x_2+\eta_b(x_3+x_4)-1,\\
&X_{pc}=\eta_a(x_1+x_2)+x_3+x_4,\\
&X_t=\exp(-i\psi_3)\{x_1\exp(i\psi_1)+x_2\exp(-i\psi_1)+\eta_b[x_3\exp(-i\psi_2)\\
&\qquad +x_4\exp(i\psi_2)]\},\\
&X_{nl}=\exp(-i\psi_4)\{\eta_a[x_1\exp(i\psi_1)+x_2\exp(-i\psi_1)]+x_3\exp(-i\psi_2)\\
&\qquad +x_4\exp(i\psi_2)\},\\
&Y_a^+=m_a y_1, \qquad Y_a^-=m_a y_2,\\
&Y_b^+=m_b y_3, \qquad Y_b^-=m_b y_4,\\
&Y_r=m_a^2(y_1+y_2)+\eta_b m_b^2(y_3+y_4)-1,\\
&Y_{pc}=\rho^{-1}[\eta_a m_a^2(y_1+y_2)+m_b^2(y_3+y_4)],\\
&Y_t=\exp(-i\psi_3)\{m_a^2[y_1\exp(i\psi_1)+y_2\exp(-i\psi_1)]+\eta_b m_b^2\{y_3\exp(-i\psi_2)\\
&\qquad +y_4\exp(i\psi_2)]\},\\
&Y_{nl}=\rho^{-1}\exp(-i\psi_4)\{\eta_a m_a^2[y_1\exp(i\psi_1)+y_2\exp(-i\psi_1)]\\
&\qquad +m_b^2[y_3\exp(-i\psi_2)+y_4\exp(i\psi_2)]\}.
\end{aligned}
\tag{62}
$$

Here, the parameters x_i and y_i for s and p-waves respectively are solutions of the linear sets.

$$
\mathbf{P}\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{bmatrix}=\begin{bmatrix} 2u\\ 0\\ 0\\ 0 \end{bmatrix}, \qquad \mathbf{Q}\begin{bmatrix} y_1\\ y_2\\ y_3\\ y_4 \end{bmatrix}=\begin{bmatrix} 2u^2\\ 0\\ 0\\ 0 \end{bmatrix}, \tag{63}
$$

where the matrices **P** and **Q** are given by

$$
\mathbf{P}=\begin{bmatrix}
u-m_1 & u+m_1 & \eta_b(u-m_2) & \eta_b(u+m_2) \\
\eta_a(m_\mathrm{p}-m_1) & \eta_a(m_\mathrm{p}+m_1) & m_\mathrm{p}-m_2 & m_\mathrm{p}+m_2 \\
-(m_1+u)\exp(\mathrm{i}\psi_1) & (m_1-u)\exp(-\mathrm{i}\psi_1) & -\eta_b(m_2+u)\exp(-\mathrm{i}\psi_2) & \eta_b(m_2-u)\exp(\mathrm{i}\psi_2) \\
-\eta_a(m_1+m_\mathrm{p})\exp(\mathrm{i}\psi_1) & \eta_a(m_1-m_\mathrm{p})\exp(-\mathrm{i}\psi_1) & -(m_2+m_\mathrm{p})\exp(-\mathrm{i}\psi_2) & (m_2-m_\mathrm{p})\exp(\mathrm{i}\psi_2)
\end{bmatrix} \tag{64}
$$

$$
\mathbf{Q}=\left[\begin{matrix}
m_1(m_1-u)-(1-u^2)(m_a^2-1) & m_1(m_1+u)-(1-u^2)(m_a^2-1) \\
\eta_a\{m_1(m_1-m_\mathrm{p})-(1-u^2)[(m_a/\rho)^2-1]\} & \eta_a\{m_1(m_1+m_\mathrm{p})-(1-u^2)[(m_a/\rho)^2-1]\} \\
\exp(\mathrm{i}\psi_1)\{m_1(m_1+u)-(1-u^2)[m_a^2-1]\} & \exp(-\mathrm{i}\psi_1)[m_1(m_1-u)-(1-u^2)(m_a^2-1)] \\
\eta_a\exp(\mathrm{i}\psi_1)\{m_1(m_1+m_\mathrm{p})-(1-u^2)[(m_a/\rho)^2-1]\} & \eta_a\exp(-\mathrm{i}\psi_1)\{m_1(m_1-m_\mathrm{p})-(1-u^2)[(m_a/\rho)^2-1]\}
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
\eta_b[m_2(m_2-u)-(1-u^2)(m_b^2-1)] & \eta_b[m_2(m_2+u)-(1-u^2)(m_b^2-1)] \\
m_2(m_2-m_\mathrm{p})-(1-u^2)[(m_b/\rho)^2-1] & m_2(m_2+m_\mathrm{p})-(1-u^2)[(m_b/\rho)^2-1] \\
\eta_b\exp(-\mathrm{i}\psi_2)\{m_2(m_2+u)-(1-u^2)[m_b^2-1]\} & \eta_b\exp(\mathrm{i}\psi_2)[m_2(m_2-u)-(1-u^2)(m_b^2-1)] \\
\exp(-\mathrm{i}\psi_2)\{m_2(m_2+m_\mathrm{p})-(1-u^2)[(m_b/\rho)^2-1]\} & \exp(\mathrm{i}\psi_2)\{m_2(m_2-m_\mathrm{p})-(1-u^2)[(m_b/\rho)^2-1]\}.
\end{matrix}\right] \tag{65}
$$

The sets of equations from equation (63) can readily be solved analytically, but the resulting expressions are lengthy and, in turn, not transparent. Numerically, one solves the sets directly, rather than inverting the matrices **P** and **Q**.

8. Special case

Although the results of the previous section apply to any situation, in many practical cases the solution can be simplified considerably because of restrictions on the order of magnitude of various parameters. It is elucidating to work out a special limit in order to reveal the fundamental structure of the Fresnel coefficients. If the incident field is a visible narrow-bandwidth laser and exactly on resonance with $\bar{\omega}$, then the relative detuning is of the order of $|\rho-1|\approx 10^{-8}$. Furthermore, the nonlinear coupling parameter γ_0 has an order of magnitude of 10^{-3}–10^{-6}, even for very strong c.w. pump fields. In this section we consider the limit $\gamma_0\to 0$ (weak interaction) and $\rho\to 1$ (resonance), which implies $\gamma_0 \ll |u^2|$ and $|\rho-1|\ll|u^2|$. First we expand the matrices **P** and **Q** up to leading order in $\rho-1$ and γ_0, after that we take the limits. For this situation the amplitude factors are related by

$$\eta_b = -\eta_a^*, \tag{66}$$

and η_a equals

$$\eta_a = -\exp(\mathrm{i}\phi)\left\{\frac{1-\rho}{\gamma_0}+\delta\left[1+\left(\frac{1-\rho}{\gamma_0}\right)^2\right]^{1/2}\right\}, \tag{67}$$

which is not necessarily a small parameter. Equation (66) expresses that the coupling strength between the two fields of the *a* solution equals the coupling strength between the two components of the *b* solution. This must be so in this limit, since $\rho\approx 1$, $\gamma\to 0$ implies $k_a^2\approx k_b^2$ and consequently the two branches of the dispersion relation coincide.

8.1. *Travelling waves*

For $0\leqslant u\leqslant 1$ the incident field is a travelling wave. In lowest order we find for the relative wavenumbers

$$m_{\mathrm{p}}=m_1=m_2=-u, \qquad m_a=m_b=1. \tag{68}$$

and the Fresnel coefficients are found to be

$$X_a^- = Y_a^- = X_b^- = Y_b^- = X_{\mathrm{r}} = Y_{\mathrm{r}} = X_{\mathrm{nl}} = Y_{\mathrm{nl}} = 0, \tag{69}$$

$$X_a^+ = Y_a^+ = \frac{1}{1+|\eta_a|^2\exp[\mathrm{i}(\psi_1+\psi_2)]},$$

$$X_b^+ = Y_b^+ = \frac{-\eta_a}{\exp[-\mathrm{i}(\psi_1+\psi_2)]+|\eta_a|^2},$$

$$X_{\mathrm{t}} = Y_{\mathrm{t}} = \exp[-\mathrm{i}(\psi_2+\psi_3)]\frac{1+|\eta_a|^2}{\exp[-\mathrm{i}(\psi_1+\psi_2)]+|\eta_a|^2},$$

$$X_{\mathrm{pc}} = Y_{\mathrm{pc}} = \eta_a\frac{\exp[-\mathrm{i}(\psi_1+\psi_2)]-1+|\eta_a|^2\{1-\exp[\mathrm{i}(\psi_1+\psi_2)]\}}{\{1+|\eta_a|^2\exp[\mathrm{i}(\psi_1+\psi_2)]\}\{\exp[-\mathrm{i}(\psi_1+\psi_2)]+|\eta_a|^2\}}. \tag{70}$$

This limit has several remarkable features. First, the Fresnel coefficients for s and p waves are identical, and therefore the scattering process is polarization independent. Second, the specularly-reflected wave disappears, and hence the field which is reflected back into the region $z>0$ consists entirely of the phase-conjugated signal with respect to the incident beam (times a factor). Third, the structure of X_{pc} is completely determined by phase factors, in the combination

$$\psi_1+\psi_2=-2\pi d(m_1-m_2). \tag{71}$$

According to equation (68), we have $m_1-m_2=0$ in first-order in $\rho-1$ and γ_0, which would make $\psi_1+\psi_2=0$ and thereby $X_{pc}=0$. However, m_1-m_2 is multiplied by the relative layer thickness d. For an interaction region of a few centimetres, we have $d\approx 10^5$ and $\psi_1+\psi_2\neq 0$, even in lowest order. Since m_1 and m_2 are the relative wave-numbers (of the z components) of the a and b waves in the PC respectively, we conclude that the phase-conjugated signal is brought about by constructive interference between the a and b modes of the PC.

If the incident field is in very close resonance with the setting $\bar{\omega}$ of the PC, then we have

$$\eta_a=-\delta\exp(\mathrm{i}\phi), \qquad |\eta_a|=1. \tag{72}$$

(More precisely: if $|\rho-1|\ll\gamma_0$). Under this condition the X_{pc} reduces to

$$X_{pc}=-\mathrm{i}\eta_a\tan[\tfrac{1}{2}(\psi_1+\psi_2)], \tag{73}$$

and the PC reflectivity becomes infinite for

$$\psi_1+\psi_2=(2n+1)\pi, \quad n \text{ integer}, \tag{74}$$

which is the famous resonance condition [3].

Even if $|\rho-1|$ is not much smaller than γ_0, the denominator of X_{pc}, equation (70), has still a resonance at the solution of

$$\exp[\mathrm{i}(\psi_1+\psi_2)]=-1, \tag{75}$$

which leads again to condition (74). With θ as the angle of incidence, we then find that the PC is resonant for later thickness

$$d=(n+\tfrac{1}{2})\frac{\cos\theta}{[(1-\rho)^2+\gamma_0^2]^{1/2}}, \quad n=0,1,2,\ldots. \tag{76}$$

At the resonance the value of X_{pc} is found to be

$$X_{pc}=\frac{2\eta_a}{1-|\eta_a|^2}, \tag{77}$$

and in between resonances, where $\exp[\mathrm{i}(\psi_1+\psi_2)]=1$, we have $X_{pc}=0$. This resonance behaviour is illustrated in figure 3.

An important conclusion is that when a PC is resonant for radiation under normal incidence ($\theta=0$), it is off-resonant for radiation which illuminates the surface under a finite angle. In spectroscopic applications, where the incident field is dipole radiation, all plane-wave components strike the PC at a different angle, and therefore, this device cannot operate as a perfect phase conjugator for the entire field.

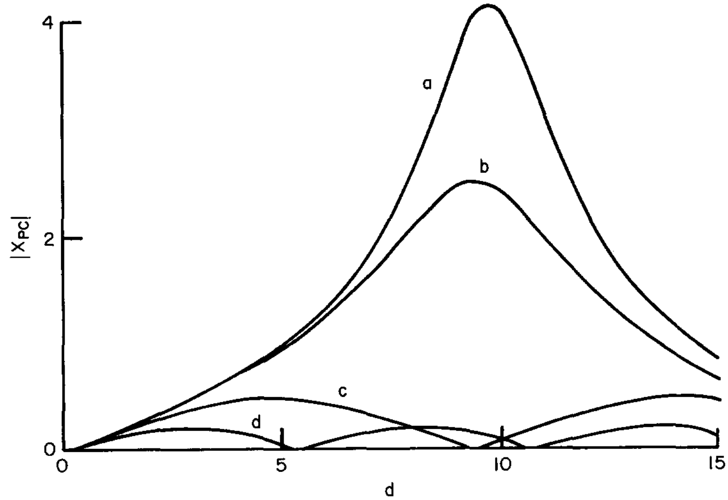

Figure 3. Absolute value of the reflection coefficient for the pc wave with s-polarization for $u=1$ and $\gamma=0{\cdot}05$, as a function of the layer thickness d. Curves a, b, c and d correspond to $\omega_1/\bar{\omega}=1{\cdot}006$, $1{\cdot}01$, $1{\cdot}05$ and $1{\cdot}1$, respectively. The peaks result from interference between a- and b waves, and the positions of the peaks are given by equation (76). For larger detunings the peaks shift and become less pronounced, in agreement with equation (77). In the limit $\omega_1=\bar{\omega}$ curve a, turns into an extremely sharp resonance.

8.2. *Evanescent waves*

For an evanescent incident wave we have

$$m_{\mathrm{p}}=-m_1=m_2=u, \tag{78}$$

and the Fresnel coefficients become

$$\begin{aligned} &X_a^-=X_b^+=X_{\mathrm{r}}=X_{\mathrm{pc}}=0,\\ &X_a^+=\frac{1}{1+|\eta_a|^2}, \qquad X_b^-=\frac{-\eta_a}{1+|\eta_a|^2},\\ &X_{\mathrm{t}}=\frac{\exp(-\mathrm{i}\psi_3)}{1+|\eta_a|^2}[\exp(\mathrm{i}\psi_1)+|\eta_a|^2\exp(\mathrm{i}\psi_2)],\\ &X_{\mathrm{nl}}=\frac{\exp(-\mathrm{i}\psi_4)}{1+|\eta_a|^2}\eta_a[\exp(\mathrm{i}\psi_1)-\exp(-\mathrm{i}\psi_2)], \end{aligned} \tag{79}$$

and the same expressions hold for p waves. We notice that there is no reflection at all back into the region $z>0$. Furthermore, if the layer thickness d is much larger than the penetration depth $1/|u|$ of the waves, then the fields in $z<-\Delta$ also vanish. We conclude that there is hardly any reflection of evanescent waves in the limit $\rho\to1$, $\gamma_0\to0$. In figure 4 we compare the Fresnel coefficients $|X_{\mathrm{pc}}|$ for travelling and evanescent waves. For evanescent waves the nonlinear interaction region is limited to the penetration depth, which is a few optical wavelengths. The PC cannot generate much radiation in such a thin layer, which explains the very small values of X_{r} and X_{pc} in this case.

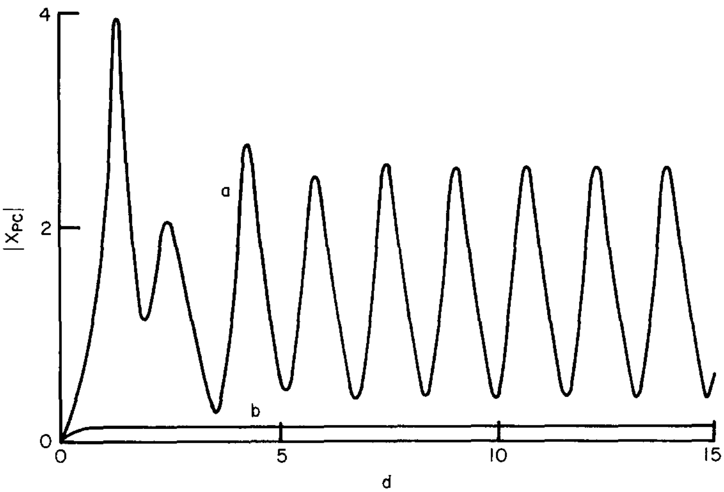

Figure 4. Reflectivity for the phase-conjugated wave as a function of d. The parameters are $\gamma_0 = 0{\cdot}05$ and $\omega_1/\bar{\omega} = 1{\cdot}01$. Plot a corresponds to a travelling incident wave ($u = 0{\cdot}25$), and curve b represents an evanescent field ($u = 0{\cdot}25\mathrm{i}$). The sharp resonances in curve a originate again from phase interference, and their position is given by equation (76). It is seen that the reflectivity for an evanescent wave is almost negligible, as compared to a travelling wave.

9. Resonances

In the previous section we found that $|X_{\mathrm{pc}}|$ acquires extreme values if the dimensionless layer thickness d is related to the angle of incidence θ, the detuning ρ and the coupling parameter γ_0 according to equation (76). These resonances appear due to perfect phase matching of the a- and b waves in the PC, as expressed by equation (75). For $|\rho - 1| \ll \gamma_0$, we have $|\eta_a| = 1$ and then $|X_{\mathrm{pc}}|$ becomes infinite, as follows from equation (77). Beside these interference resonances, a PC has a different kind of resonances which appear if we allow the angle of incidence to be non-zero and the waves to become evanescent. If we solve equation (63) for the eight parameters $x_1, \ldots, x_4$, $y_1, \ldots, y_4$, then the general expression for every parameter is a 3×3 determinant (because of the zeros on the right-hand sides), divided by $\det(\mathbf{P})$ or $\det(\mathbf{Q})$. Resonances then occur for values of ρ, u, γ_0 and d at which $\det(\mathbf{P})$ and $\det(\mathbf{Q})$ is very small. For instance, in the limit of section 8 we have

$$\det(\mathbf{P}) = (2u)^4[\exp(-\mathrm{i}\psi_2) + |\eta_a|^2 \exp(\mathrm{i}\psi_1)][\exp(-\mathrm{i}\psi_1) + |\eta_a|^2 \exp(\mathrm{i}\psi_2)], \quad (80)$$

for travelling waves, and

$$\det(\mathbf{P}) = (2u)^4(1 + |\eta_a|^2)^2 \exp[-\mathrm{i}(\psi_1 + \psi_2)], \quad (81)$$

for evanescent waves. Then it is obvious that the right-hand side of equation (80) has a minimum if the phase-matching condition (75) holds, whereas the right-hand side of equation (81) has no pronounced minima.

Without proof we state that the second kind of resonances can appear there where one of the generated waves turns from a travelling wave into an evanescent wave, or equivalently, at the branch points of the square roots which define the z components of the wave-vectors (section 5). From the expressions of section 5 in combination with relation (57) for $k_\parallel^2$, we find that the various waves are evanescent under condition

$$\left.\begin{aligned} &\text{r wave: } u^2<0,\\ &\text{pc wave: } u^2<1-\rho^2,\\ &a \text{ waves: } u^2<1-\varepsilon,\\ &b \text{ waves: } u^2<\varepsilon-\rho^2. \end{aligned}\right\} \quad (82)$$

Then the resonances are located at $u^2=0$, $u^2=1-\rho^2$, $u^2=1-\varepsilon$ and $u^2=\varepsilon-\rho^2$. Most obvious is the case $u^2=0$, or $u=0$, for which $\det \mathbf{P}$ must be small according to equations (80) and (81). We notice that the right-hand sides of (82) depend only on γ_0 and ρ, and not on u. For fixed γ_0 and u, the resonances appear at a certain detuning ρ between ω_1 of the incident field and the PC setting frequency $\bar{\omega}$. For γ_0 and ρ fixed, we can regard the resonance conditions as an equation for u (angle of incidence or inverse penetration depth) at which $|X_{pc}|$ and the other Fresnel coefficients have sharp peaks. We notice that the positions of the resonances are independent of d, in contrast to the resonances of the previous section.

Let us take ρ and γ_0 fixed, and consider the behaviour of the Fresnel coefficients as a function of u. Then the resonances are located at $u=u_{res}$, where the u_{res}s are solutions of

$$\left.\begin{aligned} &\text{pc wave: } u_{res}^2=1-\rho^2,\\ &a \text{ waves: } u_{res}^2=1-\varepsilon,\\ &b \text{ waves: } u_{res}^2=\varepsilon-\rho^2, \end{aligned}\right\} \quad (83)$$

provided that the equations have a solution in the range of u. We have suppressed the case $u=0$, since the corresponding extrema are minima. The right-hand sides of equation (83) are real, and the range of u^2 is $-\infty<u^2\leqslant 1$. To see the physical significance of the resonance conditions, we look at $u_{res}^2=1-\rho^2$ for the pc wave, and the same picture will hold for the a- and b waves. If $1-\rho^2>1$ there is no solution and the Fresnel coefficients will vary smoothly as a function of u. For $0<1-\rho^2<1$ there is a solution with $0<u_{res}^2<1$, which implies $0<u_{res}<1$ because the u values are restricted by $0\leqslant u\leqslant 1$, and $u=iv$ with $v>0$. This situation corresponds to a travelling incident wave with $u=u_{res}$. For $u>u_{res}$ the pc wave is a travelling wave, and for $u<u_{res}$ the pc-wave is evanescent. Exactly on the transition between the two situations, the Fresnel coefficients have a sharp resonance. In the case that $1-\rho^2<0$ we write $u_{res}=iv_{res}$, and the solution is $v_{res}=(\rho^2-1)^{1/2}$. Then the incident wave is evanescent, and the pc-wave is evanescent for $v>v_{res}$ and travelling for $v<v_{res}$. We notice the remarkable fact that an evanescent incident wave can be reflected by the PC as a travelling wave.

In figures 5 and 6 we have plotted $|X_{pc}|$ for a travelling and an evanescent incident field respectively. Curves a and b correspond to $\omega_1/\bar{\omega}=1{\cdot}05$ and $\omega_1/\bar{\omega}=0{\cdot}95$, which gives $\rho=0{\cdot}905$ and $\rho=1{\cdot}105$ respectively. We chose the value 0·05 for parameter γ. Then the solutions of equation (83) are for curves a

$$\left.\begin{aligned} &\text{pc wave: } u=0{\cdot}43,\\ &a \text{ waves: } v=0{\cdot}10,\\ &b \text{ waves: } u=0{\cdot}44, \end{aligned}\right\} \quad (84)$$

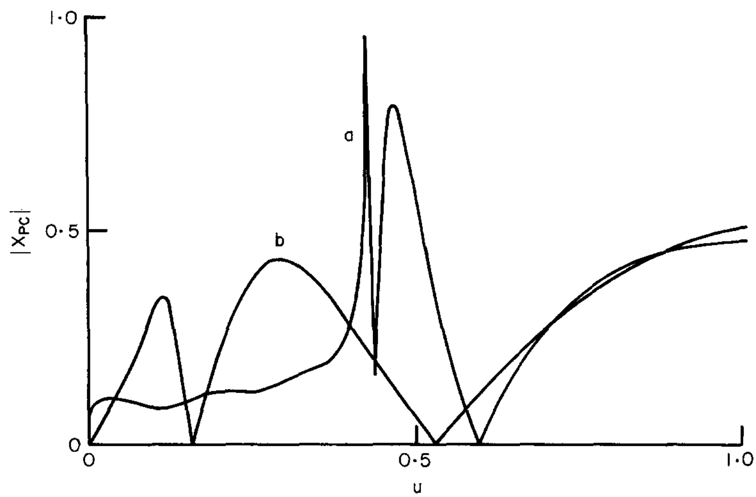

Figure 5. Absolute value of X_{pc} as a function of u for $\gamma=0{\cdot}05$, $d=5$ and $\omega_1/\bar{\omega}=1{\cdot}05$ (curve a) and $\omega_1/\bar{\omega}=0{\cdot}95$ (curve b). The two peaks in curve a are situated at $u_{res}=0{\cdot}43$ and $u_{res}=0{\cdot}44$, and they appear because the pc wave and the b wave become evanescent for lower values of u. In curve b the left-most peak corresponds to an evanescent a wave for lower values of u, and the other peak is a phase-matching interference from section 8.

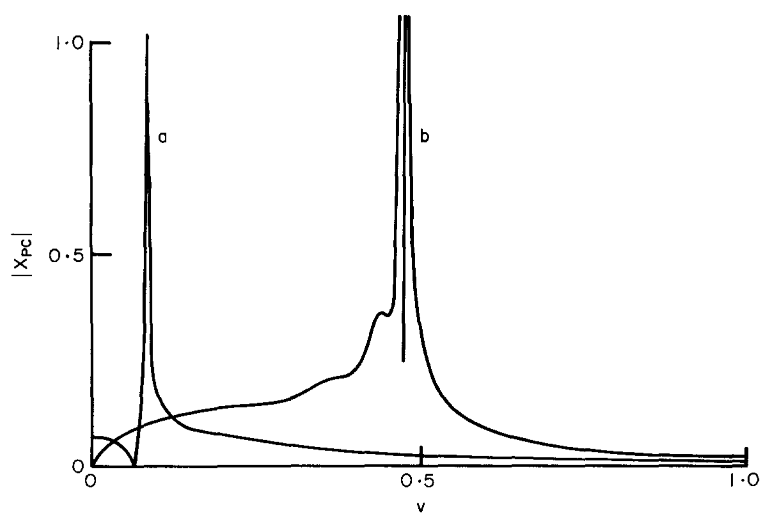

Figure 6. Same as figure 5, but now as a function of $v=-iu$. For v larger than the resonance of curve a the a wave is evanescent, and for v larger than 0·48, both the pc wave and the b wave are evanescent, which gives rise to the peaks in curve b.

and for curves *b*

$$\left.\begin{aligned} &\text{pc wave: } v = 0{\cdot}47, \\ &a \text{ waves: } u = 0{\cdot}11, \\ &b \text{ waves: } v = 0{\cdot}48. \end{aligned}\right\} \qquad (85)$$

For a travelling incident wave with $\omega_1 > \bar{\omega}$, there are two resonances as a function of u, and, for an evanescent incident wave with $\omega_1 > \bar{\omega}$, there is only a single resonance as a function of v. If $\omega_1 < \bar{\omega}$, there is only one resonance for a travelling incident wave, but two for an evanescent wave. If $\rho \approx 1$ and $\gamma \approx 0$ then the two resonances are always very close together. This can be understood from the fact that ε is very close to unity in this limit.

10. Specular reflection

In the limit of a weak interaction ($\gamma \approx 0$), in combination with close resonance ($\rho \approx 1$), the Fresnel coefficients for the specularly-reflected waves at the incident frequency ω_1 are vanishingly small (section 8). For large angles of incidence θ, the parameter $u^2 = \cos^2 \theta$ (travelling incident wave) can be of the order of γ_0 or $|\rho - 1|$, in which case the approximations of section 8 are not accurate. Since the specular wave is also reflected back into the region $z > 0$, it will interfere with the pc wave, and therefore we cannot neglect this component in the situation of grazing incidence. Figures 7 and 8 illustrate the behaviour of $|X_r|$ for various angles of incidence. For $\gamma \approx 0$ and $u \approx 1$, $|X_r|$ indeed disappears, but for $u \rightarrow 0$ the value of $|X_r|$ approaches unity. Phase matching between the *a*- and *b* waves in the PC is again responsible for the oscillatory behaviour of $|X_r|$ as a function of *d*.

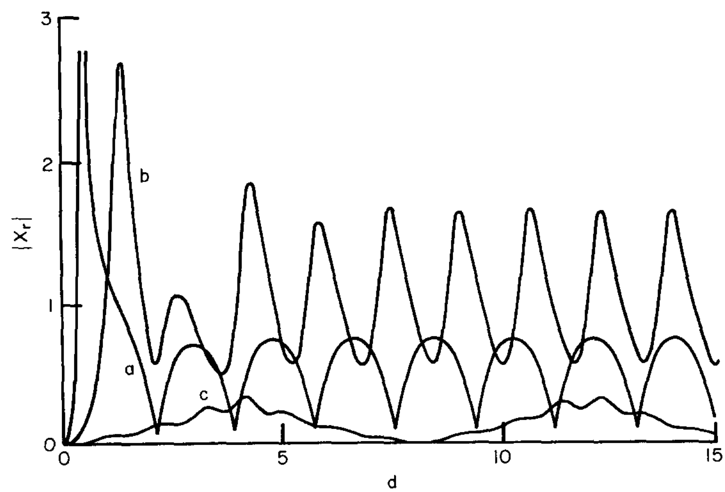

Figure 7. Reflection coefficient for the specular wave as a function of the normalized layer thickness *d*. The parameters are $\omega_1/\bar{\omega} = 1{\cdot}01$ and $\gamma = 0{\cdot}05$. For curves *a*, *b*, and *c* we took $u = 0{\cdot}2$, $0{\cdot}25$ and $0{\cdot}45$ respectively. It is seen that $|X_r|$ is not small for large angles of incidence.

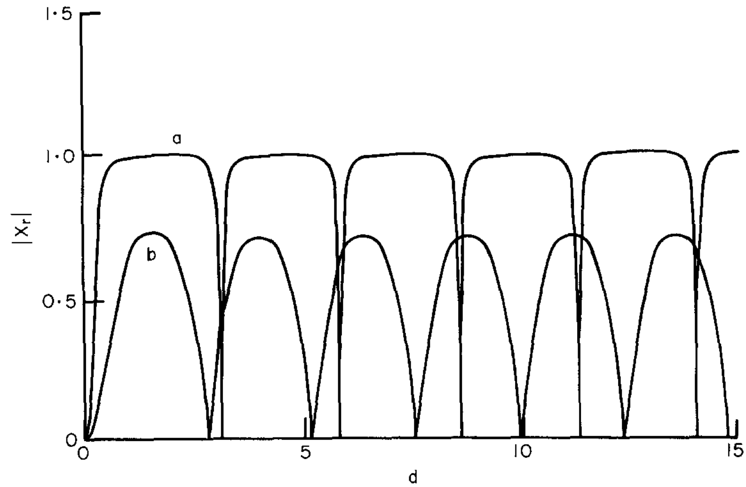

Figure 8. Same as figure 7 but with $u=0{\cdot}01$ and $0{\cdot}1$ for curves *a* and *b*, respectively. In the limit $u\to 0$, $|X_r|$ approaches unity, except at the sharp valleys, which correspond to a perfect phase mismatch between the *a*- and *b* waves.

11. Conclusions

We have studied the scattering of travelling and evanescent waves by a phase conjugator in the four-wave mixing configuration, without restrictions on the angle of incidence, the interaction strength or the frequency detuning with the pump-beams. The Fresnel coefficients for the various waves were derived from Maxwell's equations, subject to the appropriate boundary conditions, without the usual slowly-varying amplitude approximation. It was shown that in the limit of weak coupling and perfect resonance, the reflection coefficient for the pc-wave reduces to the well known result (73), which implies the resonance condition (74) for a four-wave mixing PC. We were able to track down the origin of these resonances to perfect phase matching between the two (*a* and *b*) modes of the PC. In addition, we found strong resonances at those values of the parameters for which one of the waves (*a*, *b* or pc) turns from a travelling wave into an evanescent wave. We were not able to find a convincing physical explanation for these resonances, but from numerical examples it follows that they are definitely present. Finally, we showed that for large angles of incidence the nonlinear specularly-reflected wave has an amplitude of the same order as the pc wave.

Acknowledgments

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