Confinement and redistribution of charges and currents on a surface by external fields

Henk F. Arnoldus, Daniel Jelski, and Thomas F. George
Department of Physics and Astronomy and of Chemistry, 239 Fronczak Hall, State University of New York at Buffalo, Buffalo, New York 14260

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The old problem of light scattering from a perfectly conducting surface is addressed. An electromagnetic field is incident upon the boundary, where it induces a charge and current distribution. These charges and currents emit the reflected fields. A set of equations for the charges and currents on the surface is derived by eliminating the **E** and **B** fields from Maxwell's equations with the aid of the appropriate boundary conditions. An explicit and general solution is achieved, which reveals the confinement and redistribution of the charge and the current on the surface by the external field. Expressions are obtained for the surface resolvents, or the redistribution matrices, which represent the surface geometry. Action of a surface resolvent on the incident field, evaluated at the surface, then yields the charge and current distributions. The Faraday induction appears as an additional contribution to the charge density. Subsequently, the reflected fields are expanded in spherical waves, which have the surface-multipole moments as a source. Explicit expressions are presented for the surface-multipole moments, and it is pointed out that charge conservation on the surface sets constraints on these moments. The results apply to arbitrarily shaped surfaces and to any incident field. For a specific choice of the surface structure and the external field, the solutions for the charge, the current, and the reflected fields are amenable to numerical evaluation.

I. INTRODUCTION

The study of chemistry and physics near a surface has developed rapidly during the last decade. Investigations range from classical processes like periodic deposition,[1] image formation,[2–4] and dispersion of plasmon waves[5–22] to quantum mechanical issues as Raman scattering of intense laser light,[23,24] atomic fluorescence near a rough surface,[25,26] the coupling of an atomic dipole to surface polaritons,[27] and cooperative emission processes near a conductor.[28] It appears, however, that besides these well-established theories, even the simplest problem—light scattering from an arbitrarily shaped surface—is not yet completely tractable. Early approximations like the Rayleigh–Fano expansion (neglection of multiple reflections) or the small-roughness limit provide sufficient understanding of the induced effects on a boundary by incident fields, but exact solutions in the form of general expressions for the scattered fields and the surface waves are not available at present. Contemporary closed-form solutions pertain only to polarized plane waves, incident upon gratings with well-defined geometries, like square or sinusoidal wells. The results always rely on the periodicity of the surface roughness, which implies the applicability of Fourier-series expansions, or a numerical solution of the extinction theorems, as they exist in many phrasings.[7,8,11] In this paper we consider a metallic surface, which is illuminated by an externally applied electromagnetic field with an arbitrary time dependence and spatial distribution. The surface is not assumed to be periodic, and our results apply equally well to a closed surface or to assemblies of surfaces, as for example a sphere near a grating. We achieve closed-form solutions of Maxwell's equations for the charge and current distributions on the surface and for the reflected fields, although at the expense of the assumption that the metal has a perfect conductivity.

II. THE FIELD EQUATIONS

The time development and the spatial distribution of the charge density $\rho(\mathbf{r},t)$, the current density $\mathbf{j}(\mathbf{r},t)$, the electric field $\mathbf{E}(\mathbf{r},t)$, and the magnetic field $\mathbf{B}(\mathbf{r},t)$ are governed by Maxwell's equations. If we adopt a Fourier transform of the real-valued fields

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{\pi}\,\mathrm{Re}\int_0^\infty d\omega\, \hat{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}, \tag{2.1}$$

and similarly for the other three fields, then the field equations read

$$\nabla\cdot[\epsilon(\mathbf{r})\mathbf{E}(\mathbf{r})] = \rho(\mathbf{r}), \tag{2.2}$$

$$\nabla\cdot\mathbf{B}(\mathbf{r}) = 0, \tag{2.3}$$

$$\nabla\times\mathbf{E}(\mathbf{r}) - i\omega\mathbf{B}(\mathbf{r}) = 0, \tag{2.4}$$

$$\nabla\times[\mu(\mathbf{r})^{-1}\mathbf{B}(\mathbf{r})] + i\omega\epsilon(\mathbf{r})\mathbf{E}(\mathbf{r}) = \mathbf{j}(\mathbf{r}), \tag{2.5}$$

where we have simplified the notation by writing $\mathbf{E}(\mathbf{r})$ rather than $\hat{\mathbf{E}}(\mathbf{r},\omega)$. The frequency dependence of the fields and of $\epsilon(\mathbf{r})$ and $\mu(\mathbf{r})$ will be suppressed throughout this paper.

We shall suppose that the entire space is occupied by two kinds of media, perfect conductors and perfect insulators, which are separated by boundaries. The set of all boundaries will then be referred to as the surface. Within each medium the dielectric constant $\epsilon(\mathbf{r})$ and the permeability $\mu(\mathbf{r})$ will be assumed to be $\mathbf{r}$ independent, but across the surface $\epsilon(\mathbf{r})$ and $\mu(\mathbf{r})$ are discontinuous. Conductors are specified by a relation like $\mathbf{j}(\mathbf{r}) = \gamma\mathbf{E}(\mathbf{r})$, $\gamma > 0$, and the assumption of perfect conductivity implies the limit $\gamma\to\infty$.

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Since the current density $\mathbf{j}(\mathbf{r})$ should remain finite, we obtain $\mathbf{E}(\mathbf{r}) = \mathbf{0}$ everywhere in the conductor. From Eq. (2.2) we then find $\rho(\mathbf{r}) = 0$, and Eq. (2.4) yields $\mathbf{B}(\mathbf{r}) = \mathbf{0}$, under the restriction $\omega \neq 0$. In this paper we will exclude the trivial static case $\omega = 0$. Finally, Eq. (2.5) gives $\mathbf{j}(\mathbf{r}) = \mathbf{0}$, and hence Maxwell's equations in the conductor reduce to

$$\mathbf{E}(\mathbf{r}) = \mathbf{0}, \quad \mathbf{B}(\mathbf{r}) = \mathbf{0}, \quad \rho(\mathbf{r}) = 0, \quad \mathbf{j}(\mathbf{r}) = 0. \tag{2.6}$$

Around a point $\mathbf{r}$ on the surface the fields are discontinuous. Application of Gauss' theorem on (2.2) and (2.3) and of Stokes' theorem on (2.4) and (2.5) enables us to rewrite the equations in the vicinity of the surface as

$$\mathbf{E}(\mathbf{r}^+) = \epsilon^{-1}\sigma(\mathbf{r})\mathbf{n}(\mathbf{r}), \tag{2.7}$$

$$\mathbf{B}(\mathbf{r}^+) = \mu\mathbf{i}(\mathbf{r})\times\mathbf{n}(\mathbf{r}). \tag{2.8}$$

Here $\sigma(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})$ are the surface charge and current density, respectively, and $\mathbf{n}(\mathbf{r})$ represents the unit normal vector in $\mathbf{r}$ on the surface, with the convention that it points from the conductor to the dielectric. We have introduced the notation $\mathbf{r}^+$ to indicate a point in the dielectric and close to $\mathbf{r}$. Explicitly, we write

$$\mathbf{r}^+ = \mathbf{r} + \mathbf{n}(\mathbf{r})\delta \quad \text{with } \delta\downarrow 0. \tag{2.9}$$

We note that Eqs. (2.7) and (2.8) combine the four Maxwell equations in $\mathbf{r}$ on the surface, and that they contain four unknown fields.

The dielectric is presumed to exhibit no conductivity at all, so it can be specified by $\mathbf{j} = \gamma\mathbf{E}$ with $\gamma\to 0$. This implies $\mathbf{j} = \mathbf{0}$, and from charge conservation $(\nabla\cdot\mathbf{j} = i\omega\rho)$ we find $\rho = 0$, since we required $\omega\neq 0$. Hence, all charges and currents, if any, are situated on the surface as $\sigma(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})$. The electric and magnetic fields in the dielectric are generated by $\sigma(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})$, and they contain the incident fields. This notion allows us to write Maxwell's equations for a point $\mathbf{r}$ in the dielectric as

$$\rho(\mathbf{r}) = 0, \tag{2.10}$$

$$\mathbf{j}(\mathbf{r}) = 0, \tag{2.11}$$

$$\mathbf{E}(\mathbf{r}) - \mathbf{E}(\mathbf{r})^{\text{inc}} = \frac{-1}{4\pi\epsilon}\int dA'\,\sigma(\mathbf{r}')\nabla G(\mathbf{r},\mathbf{r}') + \frac{i\omega\mu}{4\pi}\int dA'\,\mathbf{i}(\mathbf{r}')G(\mathbf{r},\mathbf{r}'), \tag{2.12}$$

$$\mathbf{B}(\mathbf{r}) - \mathbf{B}(\mathbf{r})^{\text{inc}} = \frac{-\mu}{4\pi}\int dA'\,\mathbf{i}(\mathbf{r}')\times\nabla G(\mathbf{r},\mathbf{r}'), \tag{2.13}$$

where the integrals run over the complete surface. This representation involves the Green's function of the wave equation,

$$G(\mathbf{r},\mathbf{r}') = |\mathbf{r}-\mathbf{r}'|^{-1}\exp(ik|\mathbf{r}-\mathbf{r}'|), \tag{2.14}$$

and its gradient

$$\nabla G(\mathbf{r},\mathbf{r}') = (\mathbf{r}-\mathbf{r}')|\mathbf{r}-\mathbf{r}'|^{-3}(ik|\mathbf{r}-\mathbf{r}'|-1)\times\exp(ik|\mathbf{r}-\mathbf{r}'|), \tag{2.15}$$

which contain the wave number $k = (\epsilon\mu)^{1/2}\omega$. We have to solve the set (2.12) and (2.13) for $\sigma(\mathbf{r})$, $\mathbf{i}(\mathbf{r})$, $\mathbf{E}(\mathbf{r})$, and $\mathbf{B}(\mathbf{r})$, and Maxwell's equations (2.7) and (2.8) on the surface can be considered as the boundary conditions.

III. ELIMINATION OF THE FIELDS

Maxwell's equations in the dielectric medium are basically two equations with four unknown fields, but we can eliminate the radiation fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ with the boundary conditions (2.7) and (2.8). To this end we take $\mathbf{r}$ in (2.12) and (2.13) as $\mathbf{r}^+$ from (2.9), and then substitute the boundary values for $\mathbf{E}(\mathbf{r}^+)$ and $\mathbf{B}(\mathbf{r}^+)$. This procedure leaves us with a set of two equations for $\sigma(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})$. The appearance of $G(\mathbf{r}^+,\mathbf{r}')$ and $\nabla_{\mathbf{r}^+}G(\mathbf{r}^+,\mathbf{r}')$ in the integrands of (2.12) and (2.13) is not convenient since it involves points $\mathbf{r}^+$, which are not situated on the surface. It will turn out to be more practical to have equations in which the Green's function connects only points of the surface, rather than a point on the surface to a point in the dielectric. However, care should be exercised in replacing $\mathbf{r}^+$ by $\mathbf{r}$, because the integrals are discontinuous across the surface. If we take the limit $\mathbf{r}^+\to\mathbf{r}$ properly (see Appendix), we obtain

$$\int dA'\,\sigma(\mathbf{r}')\nabla G(\mathbf{r}^+,\mathbf{r}') = -2\pi\sigma(\mathbf{r})\mathbf{n}(\mathbf{r}) + \int dA'\,\sigma(\mathbf{r}')\nabla G(\mathbf{r},\mathbf{r}'), \tag{3.1}$$

$$\int dA'\,\mathbf{i}(\mathbf{r}')G(\mathbf{r}^+,\mathbf{r}') = \int dA'\,\mathbf{i}(\mathbf{r}')G(\mathbf{r},\mathbf{r}'), \tag{3.2}$$

$$\int dA'\,\mathbf{i}(\mathbf{r}')\times\nabla G(\mathbf{r}^+,\mathbf{r}') = -2\pi\mathbf{i}(\mathbf{r})\times\mathbf{n}(\mathbf{r}) + \int dA'\,\mathbf{i}(\mathbf{r}')\times\nabla G(\mathbf{r},\mathbf{r}'), \tag{3.3}$$

and we observe that replacing $\mathbf{r}^+$ by $\mathbf{r}$ requires that we should add the terms $-2\pi\sigma(\mathbf{r})\mathbf{n}(\mathbf{r})$ and $-2\pi\mathbf{i}(\mathbf{r})\times\mathbf{n}(\mathbf{r})$ in Eqs. (3.1) and (3.3). It was already pointed out by Maradudin[29] that integrals of this kind appear to have a finite contribution from a single point. This feature can, however, also be regarded as resulting from the discontinuity of the fields across the surface. Critical comments on this issue have also been made by Agarwal[12] in a slightly different context. Combining everything then yields the set of equations

$$\sigma(\mathbf{r})\mathbf{n}(\mathbf{r}) = \frac{-1}{2\pi}\int dA'\,\sigma(\mathbf{r}')\nabla G(\mathbf{r},\mathbf{r}') + \frac{i\omega\epsilon\mu}{2\pi}\int dA'\,\mathbf{i}(\mathbf{r}')G(\mathbf{r},\mathbf{r}') + 2\epsilon\mathbf{E}(\mathbf{r})^{\text{inc}}, \tag{3.4}$$

$$\mathbf{i}(\mathbf{r})\times\mathbf{n}(\mathbf{r}) = \frac{-1}{2\pi}\int dA'\,\mathbf{i}(\mathbf{r}')\times\nabla G(\mathbf{r},\mathbf{r}') + 2\mu^{-1}\mathbf{B}(\mathbf{r})^{\text{inc}}, \tag{3.5}$$

for $\sigma(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})$. We can write $\sigma(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})$ in the integrands as

$$\sigma(\mathbf{r}) = \mathbf{n}(\mathbf{r})\cdot\{\sigma(\mathbf{r})\mathbf{n}(\mathbf{r})\}, \tag{3.6}$$

$$\mathbf{i}(\mathbf{r}) = \mathbf{n}(\mathbf{r})\times\{\mathbf{i}(\mathbf{r})\times\mathbf{n}(\mathbf{r})\}, \tag{3.7}$$

since $\mathbf{i}(\mathbf{r})$ is parallel to the surface, which shows that Eqs. (3.4) and (3.5) are essentially a set of equations for the vector fields $\sigma(\mathbf{r})\mathbf{n}(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})\times\mathbf{n}(\mathbf{r})$ on the surface.

Equation (3.5) for $\mathbf{i}(\mathbf{r})\times\mathbf{n}(\mathbf{r})$ has the form of an inhomogeneous Fredholm equation of the second kind, where the external field $2\mu^{-1}\mathbf{B}(\mathbf{r})^{\text{inc}}$ is the inhomogeneity. In the same fashion, Eq. (3.4) has $2\epsilon\mathbf{E}(\mathbf{r})^{\text{inc}}$ (and the current term) as an inhomogeneous part. Hence the incident fields can be regarded as the source terms of these equations. In this sense $\sigma(\mathbf{r})\neq 0$ and $\mathbf{i}(\mathbf{r})\neq \mathbf{0}$ are a result of the presence of the driving field, so the charges and the currents are confined on the surface by the field. If there is a net charge on the surface, this mechanism might also be conceived as a redistribution process. Equations (3.4) and (3.5) resemble the extinction theorem for the analogous problem of scattering of an incident field from a dielectric grating. The extinction theorem is, however, a homogeneous equation, and its solvability condition is equivalent to the dispersion relation for surface polaritons.

IV. REPRESENTATION OF THE SURFACE

Ordinary Fredholm equations are single-variable equations for a function on the complex plane, and they can be solved by an expansion of the function onto a suitable complete set. Our equations for $\sigma(\mathbf{r})\mathbf{n}(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})\times\mathbf{n}(\mathbf{r})$ are three-dimensional and surface-related equations for a vector field, so we have to modify the standard technique slightly. In order to accomplish this, we introduce spherical coordinates (r,θ,ϕ) with respect to an arbitrary origin, and we will abbreviate the direction θ,ϕ by the single variable Ω. Then the assembly of all points $\mathbf{r}$, which constitute the surface, can be represented by a set of functions $\xi(\Omega)_\lambda$. The $\xi(\Omega)_\lambda$ will indicate the distance from the origin to a point $\mathbf{r}$ on the surface, in the direction Ω, while the subscript λ accounts for the multiplicity (see Fig. 1). In this fashion, the surface is divided in regions, numbered by λ, where its shape is defined by a function $\xi(\Omega)_\lambda$, which determines uniquely the spherical coordinates $(r,\theta,\phi)=(\xi(\theta,\phi)_\lambda,\theta,\phi)$ of a point $\mathbf{r}$ in this region. The shape functions $\xi(\Omega)_\lambda$ will be assumed to be given, and therefore we can represent a point on the surface by its surface coordinates (λ,Ω) rather than by its spherical coordinates (r,Ω). We will use λ as a subscript and Ω as a variable.

The measure $dA(\Omega)_\lambda$ and the direction $\mathbf{n}(\Omega)_\lambda$ of the surface at a given point (λ,Ω) are fixed by its shape $\xi(\Omega)_\lambda$. For instance, the infinitesimal surface area at (λ,Ω) is given by

$$dA(\Omega)_\lambda = f(\Omega)_\lambda\, d\Omega, \tag{4.1}$$

with

$$f(\Omega)_\lambda = \xi(\Omega)_\lambda\left\{\xi(\Omega)_\lambda^2 + \left(\frac{\partial}{\partial\theta}\xi(\Omega)_\lambda\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial}{\partial\phi}\xi(\Omega)_\lambda\right)^2\right\}^{1/2}, \tag{4.2}$$

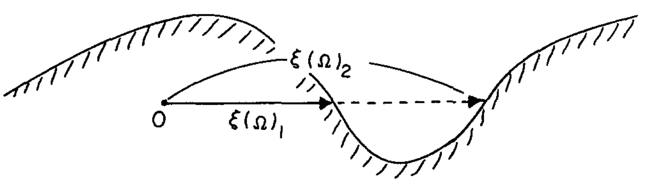

FIG. 1. Illustration of the surface multiplicity. From the origin O in the direction Ω, we find points on the surface which have a distance $\xi(\Omega)_1,\xi(\Omega)_2,\ldots$ to 0. Therefore, a description of the surface in spherical coordinates requires a set of functions $\xi(\Omega)_\lambda$.

in terms of the infinitesimal surface area $d\Omega=\sin\theta\, d\theta\, d\phi$ of the unit sphere. Hence the function $f(\Omega)_\lambda$ accounts for the deviation of the surface curvature from the curvature of a sphere, and with the aid of (4.1) we can transform a surface integral over the region λ into an integration over a part of the unit sphere. We note that not every direction Ω for a given λ corresponds to a point on the surface. It will turn out to be convenient to extend the definition (4.1) of $f(\Omega)_\lambda$ as

$$f(\Omega)_\lambda = 0, \quad \text{if } \Omega \text{ does not correspond to a point on the surface in region } \lambda. \tag{4.3}$$

Then we can write the surface integrals as

$$\int dA\cdots = \sum_\lambda \int d\Omega\, f(\Omega)_\lambda\cdots, \tag{4.4}$$

where the integrals now run over the complete unit sphere for every λ. This construction will enable us to apply the general theory of expanding vector fields on a sphere.

V. EXPANSION OF THE FIELDS

Since we are using spherical coordinates, the spherical harmonics $Y(\Omega)_{lm}$ supply a suitable complete set on the unit sphere for an expansion of the magnitude of a vector field. The direction of a vector will be expanded onto a space-fixed set of three unit vectors, denoted by $\mathbf{e}_\tau$, which is, for instance, the Cartesian set $\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z$ or the spherical set $\mathbf{e}_{+1},\mathbf{e}_0,\mathbf{e}_{-1}$. Then the vector fields $Y(\Omega)_{lm}\mathbf{e}_\tau$ constitute a complete set on the unit sphere for an expansion of an arbitrary vector field.

It is our aim to solve Eqs. (3.4) and (3.5) for $\sigma(\mathbf{r})\mathbf{n}(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})\times\mathbf{n}(\mathbf{r})$. We thus start with an expansion of these fields,

$$f(\Omega)_\lambda\sigma(\Omega)_\lambda\mathbf{n}(\Omega)_\lambda = \sum_{lm\tau} S_{lm\tau\lambda}Y_{lm}(\Omega)\mathbf{e}_\tau, \tag{5.1}$$

$$f(\Omega)_\lambda\mathbf{i}(\Omega)_\lambda\times\mathbf{n}(\Omega)_\lambda = \sum_{lm\tau} I_{lm\tau\lambda}Y_{lm}(\Omega)\mathbf{e}_\tau, \tag{5.2}$$

and note that we have included a factor $f(\Omega)_\lambda$ on the left-hand side. This is necessary, since otherwise the left-hand side of Eqs. (5.1) and (5.2) would not be properly defined for every Ω. The driving, incident fields $\mathbf{E}(\mathbf{r})^{\text{inc}},\mathbf{B}(\mathbf{r})^{\text{inc}}$ in Eqs. (3.4) and (3.5) enter only through their value on the surface, so that we can expand them on the surface set according to

$$f(\Omega)_\lambda\mathbf{E}(\Omega)_\lambda^{\text{inc}} = \sum_{lm\tau} E_{lm\tau\lambda}Y(\Omega)_{lm}\mathbf{e}_\tau, \tag{5.3}$$

$$f(\Omega)_\lambda\mathbf{B}(\Omega)_\lambda^{\text{inc}} = \sum_{lm\tau} B_{lm\tau\lambda}Y(\Omega)_{lm}\mathbf{e}_\tau. \tag{5.4}$$

The expansion coefficients for the incident fields then follow from the inverse relation

$$E_{lm\tau\lambda} = \int d\Omega\, f(\Omega)_\lambda\mathbf{E}(\Omega)_\lambda^{\text{inc}}\cdot\mathbf{e}_\tau^* Y(\Omega)_{lm}^*, \tag{5.5}$$

$$B_{lm\tau\lambda} = \int d\Omega\, f(\Omega)_\lambda\mathbf{B}(\Omega)_\lambda^{\text{inc}}\cdot\mathbf{e}_\tau^* Y(\Omega)_{lm}^*, \tag{5.6}$$

and the appearance of $f(\Omega)_\lambda$ in the integrands reflects that we actually have integrals over region λ of the surface. This illustrates that $f(\Omega)_\lambda Y(\Omega)^*_{lm}\mathbf{e}^*_\tau$ can be considered as a complete surface set for the expansion of a vector field on the surface. Note that we allow $\mathbf{e}_\tau$ to be complex, which is the case for a spherical set.

VI. THE CHARGE AND THE CURRENT DISTRIBUTIONS

It is straightfoward to rewrite Eqs. (3.4) and (3.5) for $\sigma(\mathbf{r})\mathbf{n}(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})\times\mathbf{n}(\mathbf{r})$ in terms of their expansion coefficients. We obtain

$$\sum_{l'm'\tau'\lambda'} (R^{(2)}_{lm\tau\lambda,l'm'\tau'\lambda'} - \delta_{ll'}\delta_{mm'}\delta_{\tau\tau'}\delta_{\lambda\lambda'})S_{l'm'\tau'\lambda'} = i\omega\epsilon\mu \sum_{l'm'\tau'\lambda'} R^{(3)}_{lm\tau\lambda,l'm'\tau'\lambda'}I_{l'm'\tau'\lambda'} - 2\epsilon E_{lm\tau\lambda}, \tag{6.1}$$

$$\sum_{l'm'\tau'\lambda'} (R^{(1)}_{lm\tau\lambda,l'm'\tau'\lambda'} - \delta_{ll'}\delta_{mm'}\delta_{\tau\tau'}\delta_{\lambda\lambda'})I_{l'm'\tau'\lambda'} = -2\mu^{-1}B_{lm\tau\lambda}, \tag{6.2}$$

which are two coupled inhomogeneous linear equations for the surface charge $S_{lm\tau\lambda}$ and the surface current $I_{lm\tau\lambda}$. The expansion coefficients $E_{lm\tau\lambda}$ and $B_{lm\tau\lambda}$ for the external fields are supposed to be given. The set (6.1) and (6.2) also involves three R-matrices, with matrix elements

$$R^{(1)}_{lm\tau\lambda,l'm'\tau'\lambda'} = \frac{-1}{2\pi}\int d\Omega \int d\Omega' f(\Omega)_\lambda Y(\Omega)^*_{lm} Y(\Omega')_{l'm'} \times\{\mathbf{e}^*_\tau\times(\mathbf{n}(\Omega')_{\lambda'}\times\mathbf{e}_{\tau'})\}\cdot\nabla G(\Omega,\Omega')_{\lambda\lambda'}, \tag{6.3}$$

$$R^{(2)}_{lm\tau\lambda,l'm'\tau'\lambda'} = \frac{-1}{2\pi}\int d\Omega \int d\Omega' f(\Omega)_\lambda Y(\Omega)^*_{lm} Y(\Omega')_{l'm'} \times(\mathbf{n}(\Omega')_{\lambda'}\cdot\mathbf{e}^*_{\tau'})\mathbf{e}^*_\tau\cdot\nabla G(\Omega,\Omega')_{\lambda\lambda'}, \tag{6.4}$$

$$R^{(3)}_{lm\tau\lambda,l'm'\tau'\lambda'} = \frac{-1}{2\pi}\int d\Omega \int d\Omega' f(\Omega)_\lambda Y(\Omega)^*_{lm} Y(\Omega')_{l'm'} \times\mathbf{e}^*_\tau\cdot(\mathbf{n}(\Omega')_{\lambda'}\times\mathbf{e}_{\tau'})G(\Omega,\Omega')_{\lambda\lambda'}, \tag{6.5}$$

where we have written $G(\Omega,\Omega')_{\lambda\lambda'}$ for the Green's function, which connects the points (λ,Ω) and (λ',Ω') of the surface. We emphasize that these R-matrices depend only on the geometry of the surface, and not on the external fields. Prescription of the shape of the surface determines the R-matrices. Recall, however, that the R-matrices depend on the frequency ω through the Green's function, but this is merely a parametric dependence and independent of the external field.

The expansion coefficients $S_{lm\tau\lambda}$ can always be arranged in a one-dimensional array, considered as a vector, and similarly $R^{(1)}$, $R^{(2)}$, and $R^{(3)}$ can be regarded as two-dimensional matrices. Then we can write (6.1) and (6.2) as

$$(R^{(2)} - 1)S = i\omega\epsilon\mu R^{(3)}I - 2\epsilon E, \tag{6.6}$$

$$(R^{(1)} - 1)I = -2\mu^{-1}B, \tag{6.7}$$

where we have also adopted a vector representation for the driving fields. The solution of (6.6) and (6.7) is immediately found to be

$$S = \frac{2\epsilon}{1-R^{(2)}}\left\{E - i\omega R^{(3)}\frac{1}{1-R^{(1)}}B\right\}, \tag{6.8}$$

$$I = \frac{2\mu^{-1}}{1-R^{(1)}}B, \tag{6.9}$$

which expresses the charge density S and the current density I explicitly in the externally applied fields E and B and the surface-shape matrices $R^{(1)}$, $R^{(2)}$, and $R^{(3)}$.

For vanishing external fields, e.g., $E=0$ and $B=0$, the charge and current distributions also vanish, as can be seen explicitly from Eqs. (6.8) and (6.9). Hence the charges and currents are indeed confined to the surface by the external fields. Remember that we have excluded the static case $\omega=0$, for which we can have charges on a surface without external fields. Furthermore, we can identify the resolvents $(1-R^{(2)})^{-1}$ and $(1-R^{(1)})^{-1}$ as the operators that account for the redistribution of the charges and currents, respectively, as resulting from the Lorentz force between charges and between currents. The coupling of charges and currents, which is the Faraday induction, is incorporated in the $R^{(3)}$-matrix.

VII. THE REFLECTED FIELDS

The incident field induces charges and currents on the surface, and these oscillating charges and currents emit radiation, which are the reflected fields. In this section we express these fields in terms of the expansion coefficients $S_{lm\tau\lambda}$ and $I_{lm\tau\lambda}$, as they are given explicitly in the previous section.

In Eqs. (2.12) and (2.13) we expressed the reflected electric field $\mathbf{E}(\mathbf{r}) - \mathbf{E}(\mathbf{r})^{\rm inc}$ in terms of $\sigma(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})$, and similarly $\mathbf{B}(\mathbf{r}) - \mathbf{B}(\mathbf{r})^{\rm inc}$ in terms of $\mathbf{i}(\mathbf{r})$. With (3.6) and (3.7) we can rewrite these equations in a way that $\sigma(\mathbf{r})\mathbf{n}(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})\times\mathbf{n}(\mathbf{r})$ are the source fields, and then we can apply (5.1) and (5.2) in order to find an expansion on the spherical set. However, the resulting expressions are not transparent, since they will involve the Green's function and its gradient. In order to achieve a more comprehensible result, we expand the Green's function on the spherical set. We write[30]

$$G(\mathbf{r},\mathbf{r}'_\lambda) = 4\pi ik\sum_{lm} h^{(1)}(k\xi(\Omega')_\lambda)_l Y(\Omega')_{lm} j(kr)_l Y(\Omega)^*_{lm}, \tag{7.1}$$

where $h^{(1)}_l$ and j_l are spherical Bessel functions. Here the convention is that we choose the origin of our coordinate system in the dielectric, and in such a way that the inequality

$$\xi(\Omega)_\lambda > r \tag{7.2}$$

holds for every (λ,Ω). The vector $\mathbf{r}$ is the position in the dielectric, where we wish to evaluate the reflected fields. The expansion coefficients $S_{lm\tau\lambda}$ and $I_{lm\tau\lambda}$ depend on the position of the origin, so both the charge and current distributions and the reflected fields must be evaluated with respect to the same coordinate system. Futhermore, restriction (7.2) must hold in order to apply the series expansion (7.1) of the Green's function. For a given $\mathbf{r}$, this can always be arranged.

The solution for the fields can be cast in an appealing form by the introduction of the source-term vectors

$$\mathbf{S}^{(\lambda)}_{lm} = \sum_\tau S_{lm\tau\lambda}\mathbf{e}_\tau, \tag{7.3}$$

$$\mathbf{I}_{lm}^{(\lambda)} = \sum_{\tau} I_{lm\tau\lambda}\,\mathbf{e}_{\tau}. \qquad (7.4)$$

In view of (5.1) and (5.2), these $\mathbf{S}_{lm}^{(\lambda)}$ and $\mathbf{I}_{lm}^{(\lambda)}$ are just the expansion coefficients of $f(\Omega)_\lambda \sigma(\Omega)_\lambda \mathbf{n}(\Omega)_\lambda$ and $f(\Omega)_\lambda \mathbf{i}(\Omega)_\lambda \times \mathbf{n}(\Omega)_\lambda$ after an expansion of these fields onto the set of spherical harmonics, but without a decomposition along the basis vectors $\mathbf{e}_\tau$. Futhermore, we define the vector

$$\mathbf{p}_{lm,l'm'}^{(\lambda)} = -i\int d\Omega\, f(\Omega)_\lambda\, Y(\Omega)_{lm}\, Y(\Omega)_{l'm'} \times h^{(1)}(k\xi(\Omega))_{l'}\,\mathbf{n}(\Omega)_\lambda, \qquad (7.5)$$

which is a surface integral over the region λ. It is the integrated normal vector $\mathbf{n}(\Omega)_\lambda$ times the appropriate weight functions. This vector $\mathbf{p}_{lm,l'm'}^{(\lambda)}$ depends only on the shape of the surface. After these preliminary definitions, we can write for the reflected fields

$$\mathbf{E}(\mathbf{r}) - \mathbf{E}(\mathbf{r})^{\text{inc}} = \frac{k}{\epsilon}\sum_{lml'm'\lambda} \mathbf{p}_{lm,l'm'}^{(\lambda)}\cdot\mathbf{S}_{lm}^{(\lambda)}\nabla j(kr)_{l'}\,Y(\Omega)_{l'm'}^{*} - i\omega\mu k\sum_{lml'm'\lambda} \mathbf{p}_{lm,l'm'}^{(\lambda)}\times\mathbf{I}_{lm}^{(\lambda)} j(kr)_{l'}\,Y(\Omega)_{l'm'}^{*}, \qquad (7.6)$$

$$\mathbf{B}(\mathbf{r}) - \mathbf{B}(\mathbf{r})^{\text{inc}} = \mu k\sum_{lml'm'\lambda} (\mathbf{p}_{lm,l'm'}^{(\lambda)}\times\mathbf{I}_{lm}^{(\lambda)})\nabla j(kr)_{l'}\,Y(\Omega)_{l'm'}^{*}. \qquad (7.7)$$

These explicit expressions for the fields that are emitted by the surface charge and current distributions exhibit a clear separation between the source terms $\mathbf{S}_{lm}^{(\lambda)}$ and $\mathbf{I}_{lm}^{(\lambda)}$ and the redistribution, due to the surface geometry, which is accounted for by the vector $\mathbf{p}_{lm,l'm'}^{(\lambda)}$. The spatial distribution is represented as an expansion in the spherical waves $j(kr)_l\, Y(\Omega)_{lm}^{*}$ and $\nabla j(kr)_l Y_{lm}(\Omega)^{*}$.

VIII. SURFACE MULTIPOLES

We can elucidate the significance of the expansions (7.6) and (7.7) for the reflected fields by the introduction of the surface multipoles. To this end we define the multipolar moments of the charge and the current distributions as

$$C_{lm} = \frac{k}{\epsilon}\sum_{l'm'\lambda} \mathbf{p}_{l'm',lm}^{(\lambda)}\cdot\mathbf{S}_{l'm'}^{(\lambda)}, \qquad (8.1)$$

$$\mathbf{J}_{lm} = \mu k\sum_{l'm'\lambda} \mathbf{p}_{l'm',lm}^{(\lambda)}\times\mathbf{I}_{l'm'}^{(\lambda)}, \qquad (8.2)$$

where C_{lm} is a scalar and $\mathbf{J}_{lm}$ is a vector. These multipolar moments represent the charge and current distribution of the complete surface, not just in one region λ. The emitted fields now attain the form

$$\mathbf{E}(\mathbf{r}) - \mathbf{E}(\mathbf{r})^{\text{inc}} = \sum_{lm} C_{lm}\nabla j(kr)_l Y(\Omega)_{lm}^{*} - i\omega\sum_{lm}\mathbf{J}_{lm} j(kr)_l Y(\Omega)_{lm}^{*}, \qquad (8.3)$$

$$\mathbf{B}(\mathbf{r}) - \mathbf{B}(\mathbf{r})^{\text{inc}} = \sum_{lm}\mathbf{J}_{lm}\times\nabla j(kr)_l Y(\Omega)_{lm}^{*}, \qquad (8.4)$$

which greatly resembles the multipole expansion of the fields emitted by a charge and current distribution in a restricted region of space. The distinction is of course that the source terms C_{lm} and $\mathbf{J}_{lm}$ here gain contributions from everywhere in space, rather than from a localized area. This results effectively in an exchange of the spherical Bessel function $h^{(1)}(kr)_l$ with $j(kr)_l$ in the expansion of the Green's function.

The surface multipolar moments C_{lm} and $\mathbf{J}_{lm}$ are not independent. From the fact that the fields obey Maxwell's equations, as they do by construction, it follows that they are subject to some constraints. From $\nabla\cdot(\mathbf{E}(\mathbf{r}) - \mathbf{E}(\mathbf{r})^{\text{inc}}) = 0$ we readily derive the relation

$$i\sqrt{\epsilon\mu}\,C_{lm} = \frac{\sqrt{l}}{\sqrt{2l-1}}\sum_{\mu=-(l-1)}^{l-1}\sum_{\tau}(lm1\tau|l-1\mu)\mathbf{J}_{l-1,\mu}\cdot\mathbf{e}_{\tau}^{*} + \frac{\sqrt{l+1}}{\sqrt{2l+3}}\sum_{\mu=-(l+1)}^{l+1}\sum_{\tau}(lm1\tau|l+1\mu)\mathbf{J}_{l+1,\mu}\cdot\mathbf{e}_{\tau}^{*}, \qquad (8.5)$$

for a spherical basis set $\mathbf{e}_\tau$. Here $(lm1\tau|l\pm 1\mu)$ denotes a Clebsch–Gordan coefficient. The constraint (8.5) can be considered as the surface-integrated form of charge conservation ($\nabla\cdot\mathbf{j} = i\omega\rho$) for the surface charge density $\sigma(\mathbf{r})$.

IX. CONCLUSIONS

We have studied the charge and current distributions on the boundary of a perfect conductor with a dielectric, as they are confined and redistributed there by an externally applied electromagnetic field. The surface was allowed to have an arbitrary shape, and we did not impose any periodicity condition. We obtained closed-form and exact expressions for $\sigma(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})$ everywhere on the surface. This was accomplished by deriving a set of inhomogeneous Fredholm equations of the second kind for $\sigma(\mathbf{r})\mathbf{n}(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})\times\mathbf{n}(\mathbf{r})$ from Maxwell's equations, and subsequently solving these equations by an expansion on a discrete spherical set of basis vector functions. The solution involves surface-structure matrices, the R-matrices, which are independent of the incident field. It appears that an operation of a resolvent $(1-R)^{-1}$ on the vector representation of the impinging field on the surface yields the charge and current distributions. The Faraday induction between the **E** and the **B** fields gives rise to a coupling between the equations for $\sigma(\mathbf{r})\mathbf{n}(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})\times\mathbf{n}(\mathbf{r})$, and it was accounted for by the matrix $R^{(3)}$.

Next, the structure of the fields, which are emitted by the oscillating charges and currents, was examined. The solution was cast in the form of a spherical multipolar expansion, and the multipolar moments were identified explicitly in terms of the solutions for $\sigma(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})$. The effect of the surface geometry could be incorporated entirely by the application of a surface-integrated normal-direction matrix $\mathbf{p}_{lm,l'm'}^{(\lambda)}$. In addition, it was shown that the multipolar moments for the charge and current distributions are related, which reflects the charge conservation on the surface.

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APPENDIX: DISCONTINUOUS INTEGRALS

The integrals on the right-hand sides of Eqs. (3.1) and (3.3) are discontinuous if we pass $\mathbf{r}^+$ across the surface. Therefore, care should be taken in the evaluation of the limit $\mathbf{r}^+ \to \mathbf{r}$. In this appendix we give the details of the derivation of Eq. (3.1). Then the results (3.2) and (3.3) are obtained along similar lines. The limit to be found is

$$\mathrm{Int} = \int dA'\, \sigma(\mathbf{r}') e^{ik|\mathbf{r}^+ - \mathbf{r}'|} (\mathbf{r}^+ - \mathbf{r}') \times \left\{ \frac{ik}{|\mathbf{r}^+ - \mathbf{r}'|^2} - \frac{1}{|\mathbf{r}^+ - \mathbf{r}'|^3} \right\}, \tag{A1}$$

with $\mathbf{r}^+ = \mathbf{r} + \mathbf{n}(\mathbf{r})\delta$ and $\delta \downarrow 0$. To this end we divide the surface into a small circle with radius R and around $\mathbf{r}$ and the remainder of the surface. This is illustrated in Fig. 2. For the integration over the region outside the circle, the integrand has no singularities, and we can replace $\mathbf{r}^+$ by $\mathbf{r}$. Inside the circle, however, the factor in curly brackets is singular for $\mathbf{r}^+ \to \mathbf{r}'$. This implies that we have to carry out the integration before we take the limit $\delta \downarrow 0$. This can be done as follows. First, for $\mathbf{r}'$ inside the circle we can write

$$\sigma(\mathbf{r}') \simeq \sigma(\mathbf{r}), \tag{A2}$$

$$e^{ik|\mathbf{r}^+ - \mathbf{r}'|} \simeq 1, \tag{A3}$$

since these functions vary negligibly over the singularity. Next, we write $\mathbf{r}^+ - \mathbf{r}' = (\mathbf{r} - \mathbf{r}') + \mathbf{n}(\mathbf{r})\delta$ for the vector in front of the brackets. Then we notice that the integral with $\mathbf{r} - \mathbf{r}'$ vanishes because of the cancellation of contributions from $\mathbf{b}$ and $-\mathbf{b}$ (see Fig. 2). This component disappears for every δ, and therefore also in the limit $\delta \downarrow 0$, which leaves us with

$$\mathrm{Int} = \int dA'\, \sigma(\mathbf{r}') \nabla G(\mathbf{r},\mathbf{r}') + \sigma(\mathbf{r})\mathbf{n}(\mathbf{r})\delta \int_{\substack{\text{inside}\\ \text{circle}}} dA' \left\{ \frac{ik}{|\mathbf{r}^+ - \mathbf{r}'|^2} - \frac{1}{|\mathbf{r}^+ - \mathbf{r}'|^3} \right\}. \tag{A4}$$

From Fig. 2 we see that $|\mathbf{r}^+ - \mathbf{r}'|^2 = |\mathbf{r} - \mathbf{r}'|^2 + \delta^2$. After substitution into the integrand, the integration is most easily carried out in polar coordinates, which yields

$$\delta \int_{\substack{\text{inside}\\ \text{circle}}} dA' \left\{ \frac{ik}{|\mathbf{r}^+ - \mathbf{r}'|^2} - \frac{1}{|\mathbf{r}^+ - \mathbf{r}'|^3} \right\} = 2\pi\{\tfrac{1}{2} ik\delta \log(R^2 + \delta^2) - ik\delta \log \delta + \delta/(R^2 + \delta^2)^{1/2} - 1\}. \tag{A5}$$

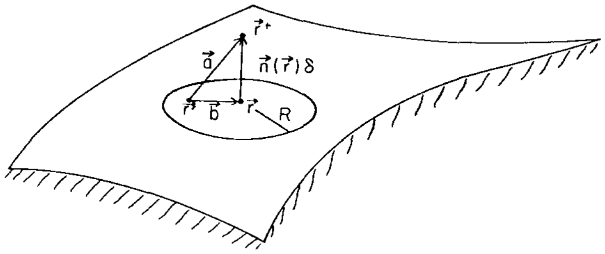

FIG. 2. Geometry for the evaluation of the limit $\mathbf{r}^+ \to \mathbf{r}$. Around $\mathbf{r}$ on the surface, we divide the surface into an infinitesimal circle of radius R, and the rest of the surface. Then the integrals are split up accordingly. The normal vector points from the surface into the dielectric and is multiplied by $\delta > 0$. We denoted $\mathbf{r}^+ - \mathbf{r}'$ by $\mathbf{a}$ and $\mathbf{r} - \mathbf{r}'$ by $\mathbf{b}$. The limit $\mathbf{r}^+ \to \mathbf{r}$ implies $R \gg \delta > 0$ and $R \to 0$. It appears that an integral over the small circle remains finite whenever the gradient of the Green's function occurs in the integrand.

In the limit $R \gg \delta > 0$ and $R \to 0$, this integral acquires the finite value of -2π, and its combination with Eq. (A4) gives expression (3.1), which was to be proved.

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